Radu P. VOINEA

Ion V. STROE     Mihai Valentin PREDOI

# TECHNICAL MECHANICS

2010

# FOREWORD

The Technical Mechanics textbook has been published by the same authors for the first time in 1996. This 2010 edition brings new examples, new figures and is correcting some typographic errors.

The publication of this edition represents also a tribute of the last two authors to the personality of one of the greatest professors that University Politehnica of Bucharest ever had, Acad. Radu P. Voinea who passed away this year.

The main structure of the initial textbook has been kept for this edition:

- The first two chapters are dedicated to fundamentals of Classical Mechanics and Physical quantities.
- The next two chapters deal with the geometry of masses: Mass centers and Moments of inertia.
- Chapters five to eight are dedicated to Statics of a material point, a rigid body and systems of rigid bodies, passing through the study of forces as sliding vectors.
- Statics of flexible cables is studied in chapter nine.
- Kinematics of a point and of a rigid body are studied in chapter ten and respectively eleven.
- Kinematics of the relative motion of a point and rigid bodies are the objects of chapter twelve, including an introduction to mechanisms analysis.
- Chapters thirteen and fourteen are dedicated to the dynamics of a material point and systems of material points.
- Dynamics of the rigid body is presented in chapter fifteen,
- Dynamics of percussive motions represent the objective if chapter sixteen.
- Dynamics of the Material Point with variable mass is briefly discussed in chapter seventeen.
- The Principles of Analytical Mechanics, Lagrange equations, Hamilton canonical equations, Poisson brackets, Variational Principles, Phase Space, The Hamilton-Jacobi partial differential equation are presented in chapters eighteen to twenty.

Technical Mechanics, being a fundamental engineering discipline, addresses students from all faculties of our university which are learning in English. All comments and suggestions of modifications are welcome.

October 2010,

Professor dr. eng. I. Stroe
Professor dr. eng. M.V. Predoi

# CONTENTS

# 1. FUNDAMENTALS OF CLASSICAL MECHANICS

## *1.1. Historical survey of Mechanics*

The concept of equilibrium and motion of material bodies is a very old one. Problems of mechanics attracted the attention of many pioneers of physics. Although the field of Classical Mechanics is based on only a few fundamental ideas, this certainly does not mean that the construction of Theoretical Mechanics was a simple matter.

Although the Babylonians and Egyptians made progress in the practical applications of Mechanics, they did not formulate general principles. It was the genius of the ancient Greeks to be particularly concerned with giving coherent account of the phenomena around them. Thales of Miletus (640-547 B.C.) postulated that the multitude of physical phenomena must be related to a single environment entity and he chooses water as his universal substance or principle. Anaximenes has chosen air. Anaximander (610-547 B.C.) deliberately avoided a name for the substance which underlies all that could be observed.

Motion was taking as a basic idea not needing further explanation. The idea of force as being the cause of motion was developed by Empedocles.

The Greek thinker who had the most lasting and dominant influence on scientific ideas was Aristotle (384 – 322 B.C.). Aristotle like many other Greek philosophers was strong in the realm of ideas, but did not in general submit his ideas to the test of experiment. Aristotle's theory of motion distinguishes between "natural" and "compulsory" motion. The former is either circular or in a strait line (free falling bodies). All other motions are compulsory and need applied forces.

Aristotle incorrectly concluded that the velocity of a falling body is proportional to its weight. About 2,000 years past before the laws of motion were established on a satisfactory basis by Galileo (1564-1642) and Newton (1642-1727). This partially demonstrates the difficulty of the problem. Galileo concluded that a force causes a change in velocity but there are no necessary forces to maintain a motion in which the magnitude and direction of the velocity dose not change. This is the basic statement of Galileo's law of inertia. He also recognized that the laws of motion are not affected by the uniform motion of a reference frame. This is the essence of Galileo's principle of relativity.

In his ***Philosophiæ Naturalis Principia Mathematica*** (the Mathematical Principles of Natural Sciences) published in 1687, Newton introduced the notion of absolute space, as a coordinate system attached to distant "fixed stars", the notion of absolute time, independent of space, the notion of mass as a positive quantity whose value does not depend on time, a measure of the quantity of matter in a body and the notion of quantity of motion as the product of the mass and the velocity of a particle. Newton formulated three laws as follows:

**First law**:
If there are no forces acting on a particle, the particle will move in a straight line with constant velocity.

**Second law:**
A force acting on a particle produces a motion in which the force is equal to the time rate of change of the quantity of motion.

**Third law:**
When two particles exert forces upon each other, the forces lie along the line joining the particles and the corresponding forces are the negative of each other.

The gravitational theory (the law stating that the force between two bodies depends on the inverse square of the mutual distance) was another of Newton's great discoveries. Newton's gravitational theory is the first example of a theory of action at distance. This theory did not give an explanation to the mechanism of gravitation, but only gave a mathematical description of the phenomenon.

The history of electricity and magnetism followed a parallel development. The discovery of electrostatics is attributed to Thales of Miletus. Later investigations on magnetism showed that the law of force between magnetic poles is similar to the Newtonian law dependence on the inverse square of the distance. However further discoveries invalidate all this. Maxwell (1831 – 1879) discovered the laws of the electromagnetic phenomena and constructed the mathematical model of the "ether", medium which allows the existence of these phenomena. Maxwell's successors tried to discover other properties of the "ether". In particular it was important to discover its motion. If it were at rest, then the "ether" would thus provide a fixed frame of reference for the whole universe, the Newtonian absolute space. The theory of "ether" replaced the theory of instantaneous action at infinite distance. All experiments designed to determine the motion gave no results since the motion of the "ether" was impossible to detect. When this conclusion was unanimously accepted, Einstein (1879- 1955) developed his revolutionary relativity theory based on two postulates:

The laws of nature (including the laws of mechanics and electrodynamics) are the same in all inertial frames.

The velocity of light in vacuum has the same value for all inertial systems, independently of the velocity the light source.

A consequence of Einstein's relativity theory is that time and space where seen no more as absolute, but as inextricably mixed not only in electromagnetism but also in mechanics and indeed in all physical phenomena . Another consequence is that relativistic mass is seen not as constant, but depending on velocity.

Max Planck (1858 – 1947) developed quantum mechanics, a valid theory for the study of systems at atomic size.
For problems of dynamics involving macroscopic bodies (with a very large number of particles, or when the number of particles is considered infinite) and velocities are very small compared to the velocity of light in vacuum the results of Classical (Newtonian) Mechanics are correct to a high order of approximation.

### *1.2. Object of Classical Mechanics*

The object of study for the Classical Mechanics, the oldest fundamental part of physics, is the motion of macroscopic systems at velocities which are very small compared to the velocity of light in vacuum.

### *1.3. Basic concepts of Classical Mechanics: space, time, mass, force*

Bodies move in space and time.

Physical **space** is reflected in Classical Mechanics by a three-dimensional Euclidean space ($E_3$). Space is assumed to be homogeneous (the same properties in each point) and isotropic (the same properties in every direction).

Physical **time** is reflected by a one-dimensional Euclidean space ($E_1$). Time is assumed to be independent of space, homogeneous, but with a unique sense of evolution.
Inertia is the tendency of a body to resist any change in its uniform motion or in its rest-still condition. This is reflected by the basic concept of **mass**, a positive scalar. The **mass** is also a measure of the capacity of a body to attract another, by gravitational force. Mass is assumed to be a constant. In fact in the relativistic mechanics, is proven that mass changes with the particle velocity as:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{1.1}$$

in which $m_0$ is the mass at rest, $v$ is the particle velocity and $c$ is the velocity of light in vacuum. However, for relatively small velocities, since $c \approx 300000$km/s, the mass can be assumed to be a constant.

The interaction between two bodies is reflected by the basic concept of **force**. A force is represented by a vector. It is determined by its magnitude (absolute value), direction (support line) and sense (positive or negative on the support line). The point of application lies on the direction line.

### *1.4. Models used in Classical Mechanics*

A modeling process is a transposition of the observed world into a mathematical form. The model is only a fiction about the real world. The principal models used in theoretical and applied mechanics are the following:

Material point, named also by some authors "particle" but not in the quantum mechanics sense, has a constant mass which is concentrated in a geometrical point. This model of body is suitable when its dimensions are small in comparison with its path.

Material line is a mathematical model for a body having two dimensions much smaller that the third. Examples of such bodies are a bar (rod, truss) and a string. A rod has, a high rigidity, while a string is very flexible.

Material surface is a mathematical model for a body having one dimension much smaller compared to the remaining two dimensions. Examples of such bodies are plates and membranes. A plate has a high rigidity, while a membrane is very flexible.

Continuous body is assumed to be indefinitely divisible without losing any of its specific properties. Three models of continuous bodies are considered in the following.

Rigid body: in spite of the action of forces, it does not show any deformations, in other words the mutual distances between the points of the body do not change. It is the main model in Classical Mechanics.

Elastic body: fully restores its original shape and dimensions after the removal of external forces or equivalent is to say the body has no residual deformations. The model will be used in several occasions in what follows.

Plastic body: does not restore its original shape and dimensions after the removal of external forces. The model is used for a particular impulsive motion or collision.

### *1.5. Principles of Classical Mechanics*

Newton stated his three famous laws or principles of Classical Mechanics using the concept of absolute space, but as it has been shown, absolute space is unobservable. Thus, the principles of classical mechanics could be reformulated as follows:

1. **There is an inertial frame, such that a material point will be relative to it either at rest, or in uniform motion along a straight line, if there are no forces acting upon it.**

Note that if there is such an inertial frame, then there are an infinite number of such inertial frames. All reference frames having a motion of translation relative to the first inertial frame are also inertial reference frames.

2. **If m denotes the mass of a material point, $\overline{a}$ the acceleration and $\overline{F}$ the force acting on the material point, then:**

$$m\overline{a} = \overline{F}\,. \qquad (1.2)$$

Note that this is not the definition of force, but a principle.

3. **If a material point A acts on a material point B with a certain force, then the material point B also acts on the material point A with a force equal in magnitude and direction, but opposite in sense**.

The forces that the material points A and B exert on each other are always directed along the straight line segment AB joining these points.
This principle is known as the principle of action and reaction. Note that these two forces are not in equilibrium because they act on different material points.

4. **If two forces $\overline{F}_1$ and $\overline{F}_2$ are acting simultaneously on the same material point, the effect they produce is the same as if the material point would be under the action of only one force $\overline{F} = \overline{F}_1 + \overline{F}_2$, the vector sum of the two forces.**

Newton considered this statement as a corollary of his three laws. In fact this is not a corollary but a fourth principle: the principle of the parallelogram. The rule of the parallelogram is a mathematical definition and it is verified experimentally. The principle of the parallelogram states that the rule of the parallelogram is true for two forces acting simultaneously on the same material point. This means that the two situations (Fig. 1, a, b) are equivalent from the mechanical point of view.

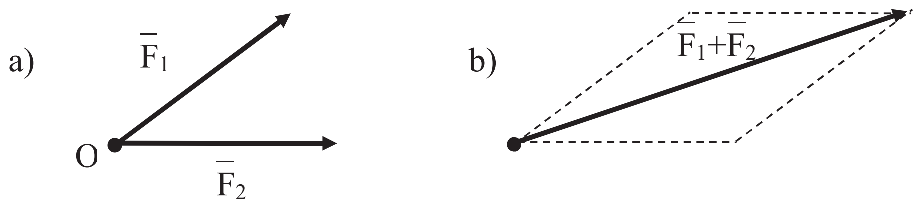


**Fig. 1. Principle of parallelogram. Two forces (a) and the sum of the two forces (b).**

### *1.6. Subdivisions of Classical Mechanics*

From the very early works, the Classical Mechanics has been subdivided into three parts:

a) **Statics** in which is studied the equilibrium of mechanical points and rigid bodies under the action of forces and moments of forces.
b) **Kinematics** in which are studied the motions of points and rigid bodies with respect to fixed and mobile systems of reference.
c) **Dynamics** in which are studied the motions of material points and rigid bodies as produced by the action of forces on these points or rigid bodies.

The first two main parts are included in the first volume of this textbook.

In the second volume is included the third part and also several chapters of **Analytic Mechanics.** This is a more recent part of Mechanics which offers the possibility to investigate, the motions of mechanical systems having several degrees of freedom, establishing in a unified manner, the differential equations of motion for linear and angular parameters defining the state of these systems.

# 2. PHYSICAL QUANTITIES

## *2.1. Physical Quantities.*

Consider a set of objects having a common physical property. The necessary and sufficient conditions to define a physical quantity corresponding to this property are the following:

- existence of a criterion making possible the distribution of these objects into equivalence classes;
- existence of a criterion to order equivalence classes;
- existence of a criterion to compare equivalence classes and to attribute to each class a scalar value (it is necessary to set up by convention two classes whose scalar values are zero and one, respectively).

When these three criteria are satisfied, a physical quantity M may be written as

$$M = n \, .u \qquad (2.1.)$$

*n* being the scalar value and *u* the measuring unit.

## *2.2. Measuring Units. International System of Measuring Units.*

The fundamental measuring units used in mechanics are the units of **length**, **time** and **mass**. By means of these units define the measuring units for other mechanical quantities. For example as measuring unit of force it will be selected a force which gives to a mass of 1 an acceleration of 1.

The International System of Measuring Units (S.I.) defines:

a) The **meter** (metre) is the unit of length. Since 1983, it is defined as the distance travelled by light in vacuum in 1⁄(299,792,458) of a second.
b) The **second** is the unit of time. Since 1967, the second has been defined to be the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium Cs 133 atom.
c) The **kilogram** is the unit of mass. The kilogram is defined as being equal to the mass of the International Prototype Kilogram (IPK) which is held at the "Bureau International des Poids et Mesures" at Sèvres near Paris, which is almost equal to the mass of one liter of water.

As it can be seen, the velocity of light in vacuum (*c*) is the constant used in defining the length:

$$c = 299\,792\,458 \quad m/s \qquad (2.2)$$

The units of the International System of Measuring Units used in Mechanics are indicated in the table 2.1.

Table 2.1. International System of Measuring Units used in Mechanics

| Physical quantity | Measuring unit | Symbol |
|---|---|---|
| length | meter | m (m) |
| time | second | s (s) |
| mass | kilogram | kg (kg) |
| force | Newton | N ($kg.m.s^{-2}$) |
| pressure | Pascal = $N/m^2$ | Pa ($kg.m^{-1}.s^{-2}$) |
| energy, work | Joule = Nm | J ($kg.m^2.s^{-2}$) |
| power | Watt = J/s | W ($kg.m^2.s^{-3}$ ) |
| velocity | meter per second | (m/s) |
| acceleration | meter per second squared | ($m/s^2$) |
| density | kilogram per cubic meter | - ($kg/m^3$) |
| frequency | Hertz | Hz ($s^{-1}$) |
| angular velocity | radian per second | rad/s ($s^{-1}$) |
| angular acceleration | radian per second squared | $rad/s^2$ ($s^{-2}$ ) |

## 2.3. Homogeneity

Each physical quantity has a dimensional equation $[a] = L^{\alpha} M^{\beta} T^{\gamma}$ , α, β, and γ being scalar values. For example a velocity $v$ has the dimensional equation $[v]=L/T=LT^{-1}$, an acceleration $[a]=L/T2=LT^{-2}$ ,a force: $[F]=[m][a]=MLT^{-2}$ and so on. A physical relation must be homogeneous i.e. if a=b or a+b=c, then physical quantities a, b and c must have the same dimensional equation:

$$[a] = [b] = [c] = L^{\alpha} M^{\beta} T^{\gamma}$$

Homogeneity is a necessary but not a sufficient condition for the correctness of a physical relation.

## 2.4. Similitude.

Let us consider a model of a real object and a physical quantity $A$ having the values $A_m$ and $A_r$ for the model and the real object, respectively. The similitude coefficient is the ratio:

$$k_A = \frac{A_r}{A_m} \tag{2.3}$$

In general $k_A \neq 1$. If it is intended to have the same value for the physical quantity $A$, on the model and on the real object, it is necessary to make a model with $k_A=1$.

# 3. MASS CENTERS

## 3.1. Static Moments

Let us consider a system of material points $A_i$ of masses $m_i$ ($i$=l,…n). By definition the static moment of the system of material points with respect to the point $O$, the origin of a Cartesian frame, is the vector:

$$\overline{S} = \sum_i m_i \overline{r}_i , \tag{3.1}$$

where $\overline{r}_i = \overline{OA_i}$ are position vectors in this frame. The static moments of the system of material points with respect to planes $Oyz$, $Ozx$ and $Oxy$ are:

$$S_{Oyz} = \sum_i m_i x_i ; \quad S_{Ozx} = \sum_i m_i y_i ; \quad S_{Oxy} = \sum_i m_i z_i . \tag{3.2}$$

If the material points are situated in plane $Oxy$, then:

$$S_{Oy} = \sum_i m_i x_i ; \; S_{Ox} = \sum_i m_i y_i , \tag{3.3}$$

are called static moments of the system of material points with respect to axes $Oy$ and $Ox$ .

## 3.2. Mass center

By definition the **center of mass** of a system of material points is a point $C$ with respect to which the static moment of the system is equal to zero.

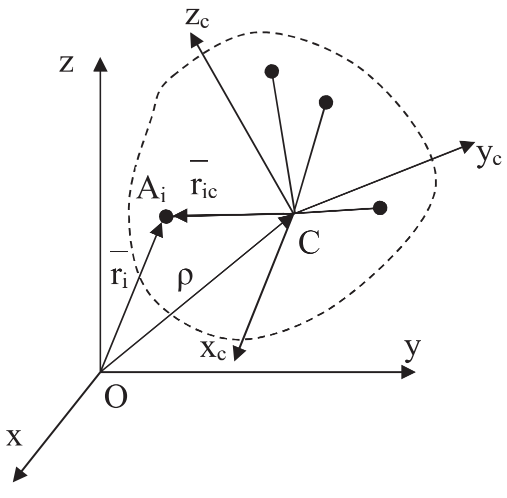


**Fig. 3.1. Position of a material point in reference frames**

Using the notations (Fig. 3.1): $\overline{OC} = \overline{\rho}$, $\overline{OA_i} = \overline{r}_i$ and $\overline{CA_i} = \overline{r}_{iC}$ it can be written successively:

$$\overline{r_i} = \overline{\rho} + \overline{r_{iC}} \quad ; \quad \sum_i m_i \overline{r_i} = \sum_i m_i \overline{\rho} + \sum_i m_i \overline{r_{iC}} . \tag{3.4}$$

According to the definition: $\sum_i m_i \overline{r_{iC}} = 0$, so that $\sum_i m_i \overline{r_i} = \sum_i m_i \overline{\rho}$. The position vector of the mass center $\overline{\rho}(\xi, \eta, \zeta)$ can thus be obtained from the last equation:

$$\overline{\rho} = \frac{\sum m_i \overline{r_i}}{\sum m_i}, \quad \text{or}$$

$$\xi = \frac{\sum m_i x_i}{\sum m_i}, \qquad \eta = \frac{\sum m_i y_i}{\sum m_i}, \qquad \zeta = \frac{\sum m_i z_i}{\sum m_i}. \tag{3.5}$$

From (3.4.) it follows that the static moment of a system of material points (with respect to a point *O*, to a plane *Oyz, Ozx* or *Oxy* (or to an axis *Oy* or *Ox*, if the material points are situated in the *Oxy* plane) is equal to the static moment with respect to the same point, plane (or axis) of the total mass placed at the center of mass.

If the material system is a continuous body, then (3.5.) becomes:

$$\overline{\rho} = \frac{\int \overline{r}\,\mathrm{dm}}{\int \mathrm{dm}}, \quad \text{or}$$

$$\xi = \frac{\int x\,\mathrm{dm}}{\int \mathrm{dm}}, \qquad \eta = \frac{\int y\,\mathrm{dm}}{\int \mathrm{dm}}, \qquad \zeta = \frac{\int z\,\mathrm{dm}}{\int \mathrm{dm}}, \tag{3.6}$$

where *dm* denotes an elementary mass.

If the continuous body is homogeneous, *dm=ρ dV* in which *ρ* is a constant mass density and *dV* is an elementary volume, then (3.6.) becomes:

$$\overline{\rho} = \frac{\iiint \overline{r}\,\mathrm{dV}}{\iiint \mathrm{dV}}, \quad \text{or}$$

$$\xi = \frac{\iiint x\,\mathrm{dV}}{\iiint \mathrm{dV}}, \qquad \eta = \frac{\iiint y\,\mathrm{dV}}{\iiint \mathrm{dV}}, \qquad \zeta = \frac{\iiint z\,\mathrm{dV}}{\iiint \mathrm{dV}}. \tag{3.7}$$

If the continuous body is a homogeneous shell or membrane $dm = \rho_s\, dA$, in which $\rho_s$ is a constant superficial mass density (mass per unit area) and *dA* is an elementary area, then (3.6.) becomes:

$$\overline{\rho} = \frac{\iint \overline{r}\,\mathrm{dA}}{\iint \mathrm{dA}}, \quad \text{or}$$

$$\xi = \frac{\iint x\,\mathrm{dA}}{\iint \mathrm{dA}}, \qquad \eta = \frac{\iint y\,\mathrm{dA}}{\iint \mathrm{dA}}, \qquad \zeta = \frac{\iint z\,\mathrm{dA}}{\iint \mathrm{dA}}. \tag{3.8}$$

If the continuous body is a homogeneous material line, $dm=\gamma\, ds$ and $\gamma$ is a constant linear mass density (mass per unit length) and $ds$ is an elementary length, then (3.6.) becomes:

$$\bar{\rho} = \frac{\int \bar{r} ds}{\int ds}, \text{ or}$$

$$\xi = \frac{\int x ds}{\int ds}, \qquad \eta = \frac{\int y ds}{\int ds}, \qquad \zeta = \frac{\int z ds}{\int ds}. \tag{3.9}$$

### *3.3. Mass center of a symmetric system of material points*

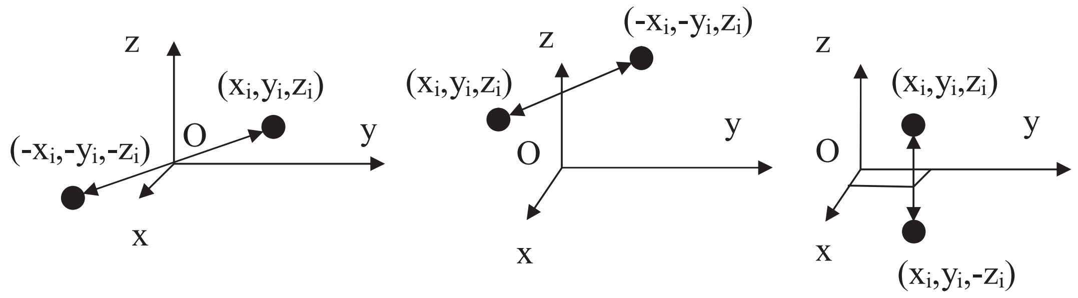


**Fig. 3.2. Symmetric material points, about a plane, an axis and a point.**

A point $O$, the line $Oz$, the plane $Oxy$ are called respectively a center, a line, or a plane of symmetry of the system of material points if for each point $A_i$ having a mass $m_i$ there exists in the system a point $B$, having the same mass $m_i$ placed symmetrically with respect to the origin $O$, line $Oz$ or respectively the plane $Oxy$.

If $O$ is the center of symmetry (Fig. 3.2 a) then for each point $A_i$ $(x_i, y_i, z_i)$ having a mass $m_i$ there exists a point $B_i$ $(-x_i, -y_i, -z_i)$ having the same mass $m_i$, and evidently $\sum m_i x_i = 0$, $\sum m_i y_i = 0$, $\sum m_i z_i = 0$. It follows by virtue of (3.5) that $\xi=0$, $\eta=0$, $\zeta=0$, therefore the mass center is $O$, the center of symmetry.

If $Oz$ is axis of symmetry for the system of material points (Fig. 3.2 b) then for each point $A_i(x_i,y_i,z_i)$ having a mass $m_i$ there exists a point $B_i$,$(-x_i,-y_i,z_i)$ having the same mass $m_i$ and evidently $\sum m_i x_i = 0, \sum m_i y_i = 0$. It follows by virtue of (3.5) that $\xi=0$, $\eta=0$, therefore the center mass lies on $Oz$ (axis of symmetry).

If $Oxy$ is the plane of symmetry of the system of material points (Fig. 3.2 c) then for each point $A_i$ $(x_i, y_i, z_i)$ having a mass $m_i$ there exists a point $B_i(x_i, y_i, -z_i)$ having

the same mass $m_i$ and evidently $\sum m_i z_i = 0$. It follows by virtue of (3.5) that $\zeta=0$, therefore the center of mass lies on the Oxy plane of symmetry.
We can conclude that a symmetric system of material points always has its center of mass either in its center of symmetry, or on its axis of symmetry, or on its plane of symmetry.

## 3.4. Mass center of systems of material points

If a system of material points $S$ may be obtained by a reunion of $n$ subsystems of material points $S_1, \ldots, S_n$ whose centers of mass are known, it can be successively written:

$$\bar{\rho} = \frac{\sum_S m_i \bar{r}_i}{\sum_S m_i} = \frac{\sum_{S_1} m_i \bar{r}_i + \ldots + \sum_{S_n} m_i \bar{r}_i}{\sum_{S_1} m_i + \ldots + \sum_{S_n} m_i} = \frac{M_1 \bar{\rho}_1 + \ldots + M_n \bar{\rho}_n}{M_1 + \ldots + M_n} \tag{3.10}$$

It follows that the center of mass of a system of material points $S$ is not altered if its subsystems $S_1, \ldots, S_n$ are replaced by material points having their masses equal to the masses $M_1, \ldots, M_n$ of the subsystems and if these equivalent material points are placed at their mass centers.
If a system of material points $S$ may be obtained by taking a system $S_2$ out of a system $S_1$, for which the mass centers are known, it can be successively written:

$$\bar{\rho} = \frac{\sum_S m_i \bar{r}_i}{\sum_S m_i} = \frac{\sum_S m_i \bar{r}_i + \sum_{S_2} m_i \bar{r}_i - \sum_{S_2} m_i \bar{r}_i}{\sum_S m_i + \sum_{S_2} m_i - \sum_{S_2} m_i} = \frac{\sum_{S_1} m_i \bar{r}_i - \sum_{S_2} m_i \bar{r}_i}{\sum_{S_1} m_i - \sum_{S_2} m_i} = \frac{M_1 \bar{\rho}_1 + (-M_2) \bar{\rho}_2}{M_1 + (-M_2)} \tag{3.11}$$

Therefore, if a subsystem of material points is taken out, its mass must be considered negative in (3.10).

## 3.5. Mass centers of some material lines, material surfaces and material bodies

### a) The arc of a circle

A material line takes shape of an arc AB (Fig. 3.3) of radius R. Because of the symmetry, the center of mass C of the arc lies on the axis $Ox$, the bisector of the central angle $2\alpha$. It follows that $\eta=0$.

If the elementary angle is taken $d\theta$, then the elementary arc is $ds = Rd\theta$ and the projection on the ox axis of the mass center of the elementary arc is $x = R\cos\theta$. It follows

$$\xi = \frac{\int xds}{\int ds} = \frac{\int_{-\alpha}^{\alpha} R\cos\theta R d\theta}{\int_{-\alpha}^{\alpha} R d\theta} = \frac{R\sin\theta\Big|_{-\alpha}^{\alpha}}{\theta\Big|_{-\alpha}^{\alpha}} = \frac{R\sin\alpha}{\alpha} \tag{3.12}$$

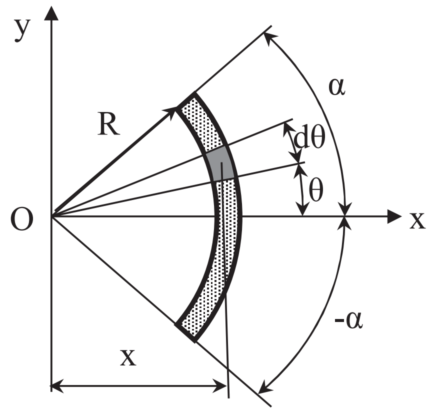


**Fig. 3.3. The mass center of an arc of a circle.**

**b) The triangle**

By cutting a triangle in infinite thin stripes (equivalent with material lines) parallel to one of its sides, the mass centers of these segments lie on the triangle *median*, and by consequence so does the center of mass of the triangle. It follows from this statement that the center of mass of the triangle is at the point of intersection of the three medians (Fig. 3.4).

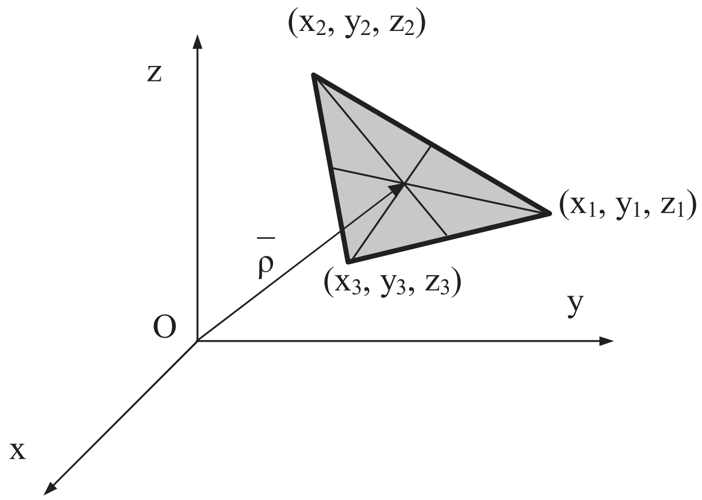


**Fig. 3.4 The mass center of a triangle (triangular plate).**

The crossing point of the medians is at a distance of one third of the corresponding distance from each side and two thirds from the corners. Considering the three corners of the triangle having the coordinates ($x_1$ $y_1$ $z_1$ ), ($x_2$ $y_2$ $z_2$ ), ($x_3$ $y_3$ $z_3$ ) it can be proven that:

$$\xi = \frac{x_1 + x_2 + x_3}{3}; \quad \eta = \frac{y_1 + y_2 + y_3}{3}; \quad \zeta = \frac{z_1 + z_2 + z_3}{3} \tag{3.13}$$

**c) The trapezium**

The centers of the segments parallel to the bases of a trapezium lay on the median EF and hence so does the center of gravity of the trapezium.

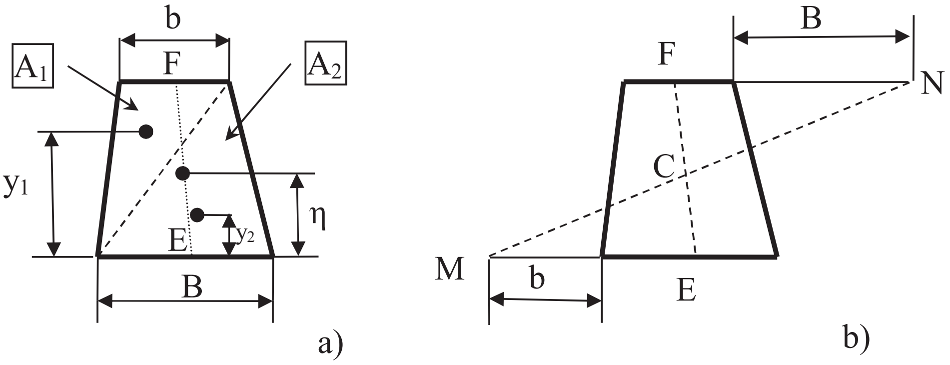


**Fig. 3.5 Mass center of a trapezium**

In order to determine the distance $\eta$ of the center of gravity C from the base, the trapezium is divided into two triangles (Fig. 3.5 a).

The areas are $A_1=bh/2$, $A_2=Bh/2$ and the mass centers are at heights $y_1=2h/3$, $y_2=h/3$, in which B and b are the lengths of the two parallel edges distanced by $h$. It follows that:

$$\eta = \frac{bh/2 \cdot 2h/3 + Bh/2 \cdot h/3}{bh/2 + Bh/2} = \frac{B+2b}{3(B+b)}h. \tag{3.14}$$

From this, follows the geometric construction of the center of gravity shown in Fig. 3.5b. From the similarity of triangles MEC and NFC one gets:

$$\frac{\eta}{h-\eta} = \frac{\frac{B}{2}+b}{B+\frac{b}{2}},$$

which confirms the above formula.

**d) The sector of a circle**

A sector of a circle is considered symmetrically placed about the Ox axis (Fig. 3.6), the bisector of the angle $2\alpha$. It follows that $\eta=0$. An elementary area is defined by radiuses $\rho$ and $\rho+d\rho$ and angles $\theta$ and $\theta+d\theta$. The elementary area is then $dA=d\rho\, \rho d\theta$, $x=\rho \cos\theta$ and

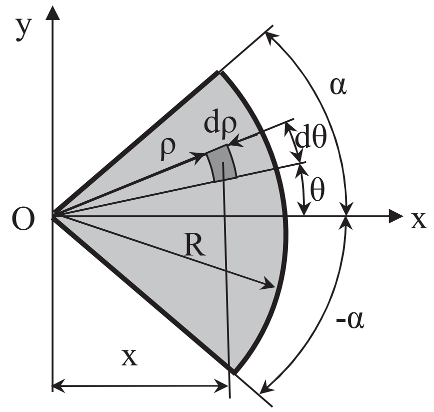


**Fig. 3.6. The mass center for the sector of a circle**

$$\xi=\frac{\int xdA}{\int dA}=\frac{\iint \rho\cos\theta\cdot\rho\cdot d\rho d\theta}{\iint \rho\cdot d\rho d\theta}=\frac{\int_{-\alpha}^{\alpha}\cos\theta\cdot d\theta\cdot\int_{0}^{R}\rho^{2}d\rho}{\int_{-\alpha}^{\alpha}d\theta\cdot\int_{0}^{R}\rho\cdot d\rho}=\frac{\sin\theta\cdot\Big|_{-\alpha}^{\alpha}\cdot\frac{\rho^{3}}{3}\Big|_{0}^{R}}{\theta\cdot\Big|_{-\alpha}^{\alpha}\cdot\frac{\rho^{2}}{2}\Big|_{0}^{R}}$$

or:

$$\xi=\frac{2R\sin\alpha}{3\alpha}. \tag{3.15}$$

For a semicircle 2α=π and the previous formula becomes:

$$\xi=\frac{4R}{3\pi}. \tag{3.16}$$

## e) **Zone of a hollow sphere**

Consider the zone of a sphere of radius R (Fig. 3.7) between parallel planes $z=z_1$, and $z=z_2$. Because of symmetry, the center of mass C of the zone lies on the axis Oz.

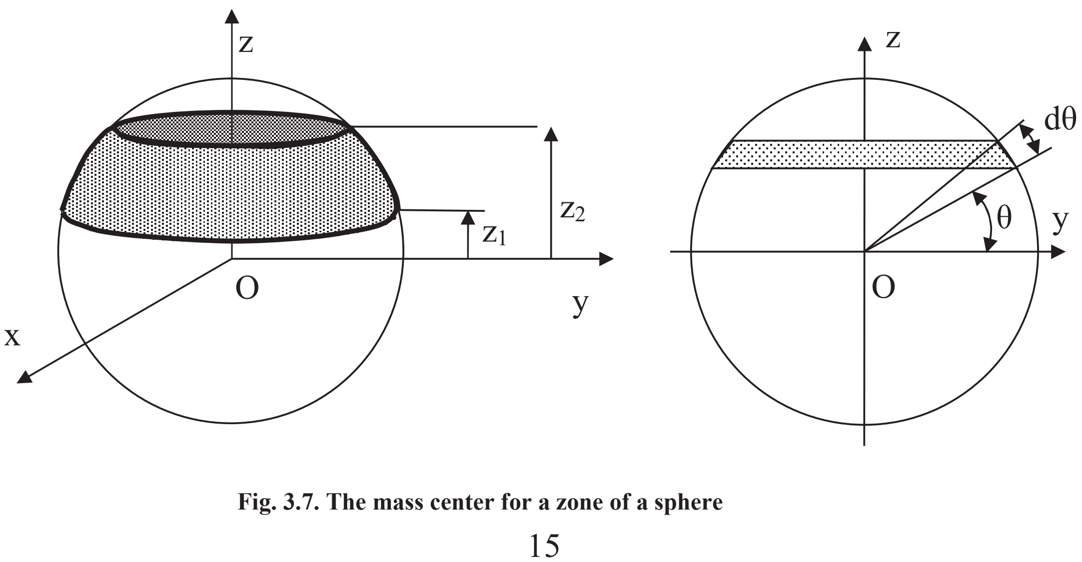


**Fig. 3.7. The mass center for a zone of a sphere**

It follows that *ξ=0, η=0*. If the elementary area is taken as a zone of the same sphere, between parallel planes $z = R\sin\theta$ and *z+dz*, of radius $r = R\cos\theta$, corresponding to angles *θ* and *θ+dθ* (Fig. 3.6) from the sphere center, then the elementary area is:

$$dA = 2\pi r\, ds = 2\pi r\, Rd\theta = 2\pi R^2 \cos\theta d\theta, \tag{3.1}$$

and the mass center is located at:

$$\xi = \frac{\int z \cdot dA}{\int dA} = \frac{\int_{\theta_1}^{\theta_2} R\sin\theta \cdot 2\pi R^2 \cos\theta \cdot d\theta}{\int_{\theta_1}^{\theta_2} 2\pi R^2 \cos\theta \cdot d\theta} = \frac{R}{2} \frac{\left.\frac{\cos 2\theta}{2}\right|_{\theta_2}^{\theta_1}}{\left.\sin\theta\right|_{\theta_1}^{\theta_2}} = \frac{R}{4} \frac{\cos 2\theta_1 - \cos 2\theta_2}{\sin\theta_2 - \sin\theta_1} \tag{3.2}$$

$$= \frac{R}{4} \frac{2\left(\sin^2\theta_2 - \sin^2\theta_1\right)}{\sin\theta_2 - \sin\theta_1} = \frac{R}{2}\left(\sin\theta_2 + \sin\theta_1\right) = \frac{z_2 + z_1}{2}$$

As a particular case, the mass center of a hemispheric dome is $\zeta = \frac{R}{2}$.

f) **The circular cone**

Consider a cone of height *h*, having the radius of the circle of the base *R* (Fig. 3.8). Because of the symmetry the center of mass of the cone lies on the axis *Oz*. It follows that *ξ=0*, *η=0*. If the elementary volume is a frustum of a cone between parallel planes *z* and *z+dz*, then the mass center of the elementary volume is at *z*, and its volume is $dV = \pi r^2 dz$ in which the radius is obtained from the proportionality relation *r/R=z/h*.

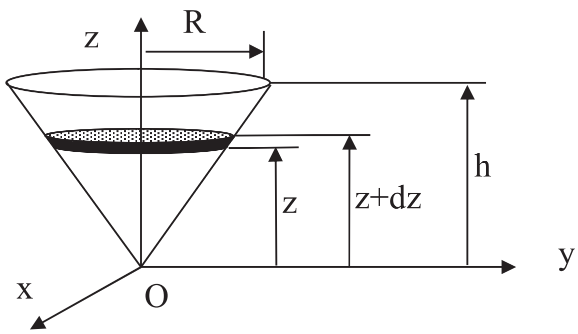


**Fig. 3.8 Mass center for a circular cone**

The mass center is

$$\xi = \frac{\int z \cdot dV}{\int dV} = \frac{\int_0^h \pi \frac{R^2}{h^2} z^3 dz}{\int_0^h \pi \frac{R^2}{h^2} z^2 dz} = \frac{\left.\frac{z^4}{4}\right|_0^h}{\left.\frac{z^3}{3}\right|_0^h} = \frac{3h}{4} \tag{3.18.}$$

Hence the mass center of a cone is independent of the base radius R. Note that the same location of the mass center is valid for a pyramid or an arbitrary cone (for example if the base is not a circle).

g) **The Hemisphere**.
Consider a hemisphere of radius R (Fig. 3.9). Because of symmetry the center of mass C of the hemisphere lies on the axis Oz.

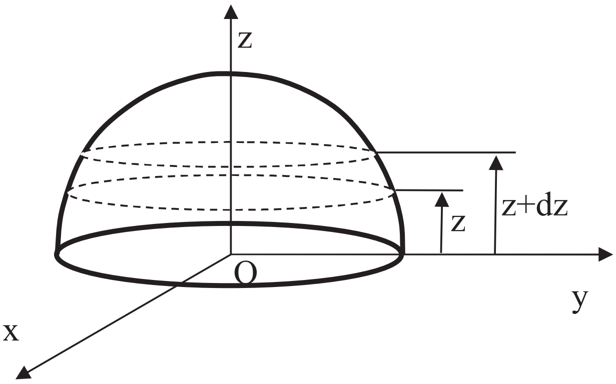


**Fig. 3.9 Mass center of a hemisphere**

It follows that $\xi=0$, $\eta=0$. The elementary volume is taken as a solid section in the hemisphere between parallel planes $z$ and $z+dz$. The $z$ coordinate of the mass center of such an elementary volume is obviously at $z$ and its volume is $dV = \pi r^2 dz = \pi\left(R^2 - z^2\right)dz$, because $r^2 = R^2 - z^2$. Thus

$$\zeta = \frac{\int z \cdot dV}{\int dV} = \frac{\int_0^R z\pi(R^2 - z^2)dz}{\int_0^R \pi(R^2 - z^2)dz} = \frac{R^2\int_0^R zdz - \int_0^R z^3 dz}{R^2\int_0^R dz - \int_0^R z^2 dz} = \frac{\frac{R^4}{2} - \frac{R^4}{4}}{R^3 - \frac{R^3}{3}} = \frac{3R}{8} \qquad (3.19)$$

h) For the plate shown in Fig. 3.10 calculate in the indicated Cartesian frame the mass center. The formulas for systems of bodies become in this case

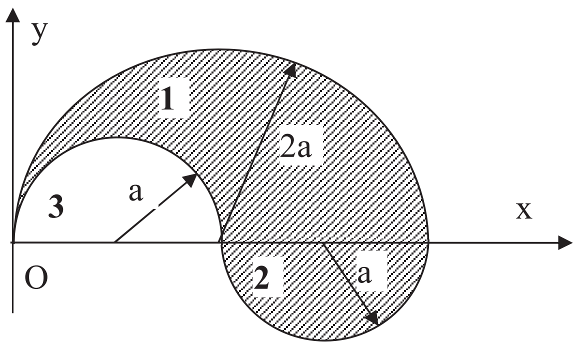


**Fig. 3.10. Mass center of a composite plate**

$$\xi = \frac{A_1\xi_1 + A_2\xi_{21} + A_3\xi_3}{A_1 + A_2 + A_3}; \quad \eta = \frac{A_1\eta_1 + A_2\eta_2 + A_3\eta_3}{A_1 + A_2 + A_3}$$

with: $A_1 = \frac{1}{2}\pi\left(2a\right)^2; A_2 = \frac{1}{2}\pi a^2; \; A_3 = -\frac{1}{2}\pi a^2$

$\xi_1=2a;$ $\quad\xi_2=a;$ $\quad\xi_3=3a;$

$$\eta_1 = \frac{4(2a)}{3\pi} = \frac{8a}{3\pi}; \quad \eta_2 = -\frac{4a}{3\pi}; \quad \eta_3 = \frac{4a}{3\pi}$$

It follows that:

$$\xi = \frac{2\pi a^2 \cdot 2a + 1/2\pi a^2 \cdot a - 1/2\pi a^2 \cdot 3a}{2\pi a^2 + 1/2\pi a^2 - 1/2\pi a^2} = 1.5a$$

$$\eta = \frac{2\pi a^2 \cdot 8a/3\pi - 1/2\pi a^2 \cdot 4a/3\pi - 1/2\pi a^2 \cdot 4a/3\pi}{2\pi a^2 + 1/2\pi a^2 - 1/2\pi a^2} = \frac{2a}{\pi}$$

### *3.6. Area and volume of a body of axial symmetry. Theorems of Guldin-Pappus.*

Computing the lateral area or the volume of a body obtained by rotating a curvilinear line (Γ) around an axis, can be of interest in technical applications.

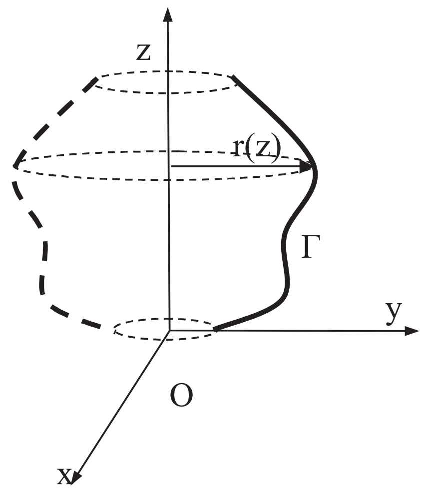


**Fig. 3.11 An axially symmetrical body.**

It can be assumed that symmetry axis is Oz and so (Γ) is defined as $r(z): z \in [0, h]$.

### 3.6.1. The first theorem of Guldin-Pappus

The lateral area of an axially symmetrical body can be obtained from the product:

$$A = 2\pi r_\Gamma L \,, \tag{3.20}$$

in which $r_\Gamma$ is the distance between the symmetry axis and the mass center of (Γ) considered as a material line.

In order to proof the theorem, it must be defined first the lateral area of an axially symmetrical body:

$$A = \int_0^h 2\pi r(z)\,dz \,. \tag{3.21}$$

The mass center of the "material line" (Γ) is

$$r_\Gamma = \frac{\int_0^L r ds}{\int_0^L ds} \,. \tag{3.22}$$

For bodies with slowly varying radiuses $ds = \sqrt{dr^2 + dz^2} = dz\sqrt{1+\left(\frac{dr}{dz}\right)^2}\,dz$, so that the last integral leads to $\int_0^h r(z)\,dz = L\,r_\Gamma$, which proves the theorem.

Example. A sphere of radius R is obtained by rotating a semicircle of the same radius around its diameter. Find the area of the sphere.

The length of the semicircle is $L = \pi R$, and the mass center lies at a distance $r_\Gamma = \frac{R\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi}R$. The area of the sphere is then $A = 2\pi\frac{2}{\pi}R\cdot\pi R = 4\pi R^2$, which confirms the known result.

### 3.6.2. The second theorem of Guldin-Pappus

The volume of an axially symmetrical body can be obtained from the product:

$$V = 2\pi r_A A \,, \tag{3.23}$$

in which $r_A$ is the distance between the symmetry axis and the mass center of (A), the area bordered by (Γ).

In order to proof the theorem, it must be defined first the volume of an axially symmetrical body:

$$V = \int_0^h \pi [r(z)]^2 \, dz \,. \tag{3.24}$$

The mass center of the "material surface" (A) is :

$$r_A = \frac{\iint_A r dA}{\iint_A dA} = \frac{\int_0^h \frac{r}{2} r dz}{A} = \frac{\int_0^h r^2 dz}{2A} \,. \tag{3.25}$$

Replacing $\int_0^h r^2 dz$ in the formula for V, the second Guldin-Pappus theorem is proven.

<u>Example</u>

A sphere of radius R is obtained by rotating a semicircle of the same radius around its diameter. Find the volume of the sphere.

The area of the semicircle is $A = \frac{\pi R^2}{2}$ and the distance between the diameter and the mass center of the "semicircular plate" is $r_A = \frac{2}{3} \frac{R \sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{4R}{3\pi}$. The volume of the sphere is thus $V = 2\pi \frac{4R}{3\pi} \frac{\pi R^2}{2} = \frac{4\pi R^3}{3}$, confirming the well known result.

# 4. MOMENTS OF INERTIA

## 4.1. Definitions

A Cartesian coordinate frame *Oxyz* and a system of material points $A_i(x_i,y_i,z_i)$ of masses $m_i$ are considered (Fig. 4.1).

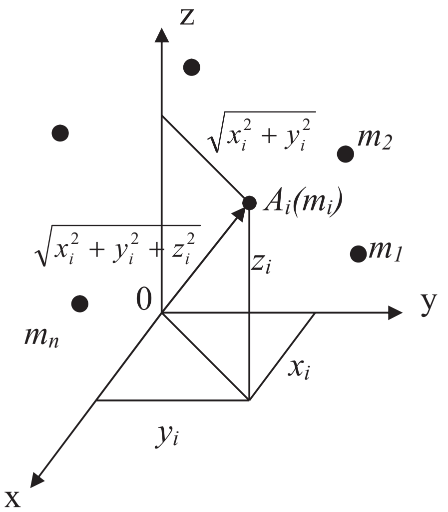


**Fig. 4.1 A system of material points**

By definition:

$$J_{yoz} = \sum_{i=1}^{n} m_i x_i^2; \quad J_{zox} = \sum_{i=1}^{n} m_i y_i^2; \quad J_{xoy} = \sum_{i=1}^{n} m_i z_i^2 , \qquad (4.1)$$

are called **moments of inertia of the system of material points with respect to the planes** *Oyz*, *Ozx* and *Oxy* respectively. In a similar manner can be defined the **moments of inertia with respect to axes** *Ox, Oy* and *Oz* as:

$$J_x = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2); \quad J_y = \sum_{i=1}^{n} m_i (z_i^2 + x_i^2); \quad J_z = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2) \qquad (4.2)$$

and the **moment of inertia with respect to the origin**:

$$J_o = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2 + z_i^2) . \qquad (4.3)$$

By definition:

$$J_{xy} = J_{yx} = \sum_{i=1}^{n} m_i x_i y_i; \quad J_{yz} = J_{zy} = \sum_{i=1}^{n} m_i y_i z_i; \quad J_{zx} = J_{xz} = \sum_{i=1}^{n} m_i x_i z_i \qquad (4.4)$$

are called **products of inertia**. These ten expressions are not independent. The following relations are verified:

$$J_{yoz} = \frac{1}{2}\left[J_y + J_z - J_x\right]; \; J_{zox} = \frac{1}{2}\left[J_z + J_x - J_y\right]; \; J_{xoy} = \frac{1}{2}\left[J_x + J_y - J_z\right];$$

$$J_O = \frac{1}{2}\left(J_x + J_y + J_z\right). \qquad (4.5)$$

It follows that only $J_x$, $J_y$, $J_z$, $J_{xy}$, $J_{yz}$, $J_{xz}$ are independent. By definition the symmetric matrix:

$$[J]=\begin{bmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{bmatrix} \tag{4.6}$$

is called the **matrix of moments of inertia** and the symmetric second – order tensor having the components $J_x$, $J_y$, $J_z$, $J_{xy}= J_{yx}$, $J_{yz}= J_{zy}$, $J_{xz}= J_{zx}$ represents the **tensor of moments of inertia**.

### *4.2. Moments of inertia and products of inertia with respect to parallel axes*

Consider a system of material points $A_i$ of masses $m_i$ (i=1,2,…,n) and two Cartesian coordinate frames *Cxyz* and *Ox'y'z'* (C is the center of mass for the given system of points and the axes of the two systems are parallel) (Fig. 4.2).

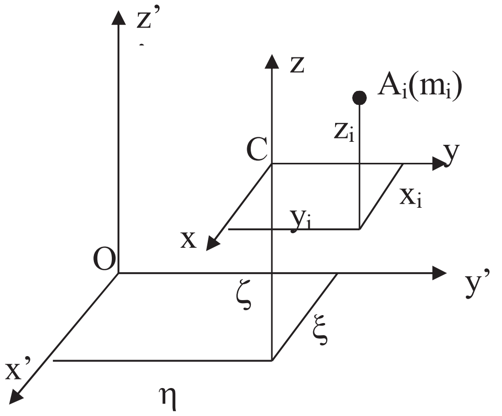


**Fig. 4.2 System of material points relative to two reference frames with parallel axes**

Denoting by $x_i, y_i, z_i$, and $x_i', y_i', z_i'$ the coordinates of $A_i$ with respect to *Cxyz* and *Ox'y'z'* respectively and by $\xi$, $\eta$, $\zeta$ the coordinates of C with respect to *Ox'y'z'* and since $x_i' = \xi + x_i$; $y_i' = \eta + y_i$; $z_i' = \zeta + z_i$ it follows that :

$$\begin{aligned} J_{x'} &= \sum_{i=1}^{n} m_i \left( y_i'^2 + z_i'^2 \right) = \sum_{i=1}^{n} m_i \left[ (\eta + y_i)^2 + (\zeta + z_i)^2 \right] \\ &= \sum_{i=1}^{n} m_i \left( y_i^2 + z_i^2 \right) + \sum_{i=1}^{n} m_i (\eta^2 + \zeta^2) + 2\eta \sum_{i=1}^{n} m_i y_i + 2\zeta \sum_{i=1}^{n} m_i z_i \\ &= J_x + M \left( \eta^2 + \zeta^2 \right) \end{aligned} \tag{4.7}$$

The mass of the system of material points is $M = \sum_{i=1}^{n} m_i$, proving that the formula is also valid for a rigid body.

$$J_{x'y'} = \sum_{i=1}^{n} m_i x_i^{'} y_i^{'} = \sum_{i=1}^{n} m_i (\xi + x_i)(\eta + y_i) = \sum_{i=1}^{n} m_i x_i y_i + \xi\eta \sum_{i=1}^{n} m_i$$
$$+\xi \sum_{i=1}^{n} m_i y_i + \eta \sum_{i=1}^{n} m_i x_i = J_{xy} + M\xi\eta \quad (4.8)$$

The following expressions were used in deducing the last two formulas, which are representing the static moments of the system of material points with respect to planes passing through the center of mass: $\sum m_i x_i = 0, \sum m_i y_i = 0, \sum m_i z_i = 0$. Analogously, one can find the other similar relations:

$$J_{x'} = J_x + M(\eta^2 + \zeta^2); \quad J_{x'y'} = J_{xy} + M\xi\eta;$$
$$J_{y'} = J_y + M(\zeta^2 + \xi^2); \quad J_{y'z'} = J_{yz} + M\eta\zeta; \quad (4.9)$$
$$J_{z'} = J_z + M(\xi^2 + \eta^2); \quad J_{z'x'} = J_{zx} + M\zeta\xi.$$

In general, if an axis Δ is passing through the center of mass of a system of material points and is parallel to the axis Δ', then

$$J_{\Delta'} = J_{\Delta} + Md^2, \quad (4.10)$$

where $d$ denotes the distance between the axis Δ and Δ', whereas $J_{\Delta}$ and $J_{\Delta'}$ denote the moments of inertia with respect to these axes. These expressions are known as **Steiner's formulas**.

### *4.3. Moments of inertia with respect to axes intersecting in a point*

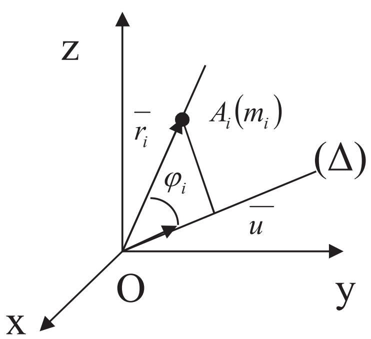


**Fig. 4.3 Distance between a point and an axis**

A system of material points $A_i$ of masses $m_i$ (i=1,2,...,n), is placed in a Cartesian coordinate system *Oxyz* and an axis Δ is passing through its origin C. If we denote by $\bar{u}$ the unit vector of Δ and by $\bar{r}_i$ the vector $\overline{OA_i}$, then the distance $d_i$ from $A_i$ to the axis Δ is given by (Fig. 4.3):

$$d_i = |\bar{r}_i| \sin\varphi_i = |\bar{r}_i \times \bar{u}|. \quad (4.11)$$

The components of the position vector $\bar{r}_i$ are denoted by $x_i$, $y_i$, $z_i$ and the components of the unit vector $\bar{u}$ are $(\cos\alpha, \cos\beta, \cos\gamma)$ so that:

$$\bar{r}_i \times \bar{u} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_i & y_i & z_i \\ \cos\alpha & \cos\beta & \cos\gamma \end{vmatrix} \quad (4.12)$$
$$= (y_i \cos\gamma - z_i \cos\beta)\bar{i} + (z_i \cos\alpha - x_i \cos\gamma)\bar{j} + (x_i \cos\beta - y_i \cos\alpha)\bar{k}$$

The distance between the arbitrary point $A_i$ and the axis Δ is:

$$d_i = \left|\overline{r_i} \times \overline{u}\right| = \sqrt{\left(y_i \cos\gamma - z_i \cos\beta\right)^2 + \left(z_i \cos\alpha - x_i \cos\gamma\right)^2 + \left(x_i \cos\beta - y_i \cos\alpha\right)^2} \tag{4.13}$$

It follows that:

$$\begin{aligned} J_\Delta &= \sum_{i=1}^{n} m_i d_i^2 \\ &= \sum_{i=1}^{n} m_i \left[\left(y_i \cos\gamma - z_i \cos\beta\right)^2 + \left(z_i \cos\alpha - x_i \cos\gamma\right)^2 + \left(x_i \cos\beta - y_i \cos\alpha\right)^2\right] \\ &= \left[\sum_{i=1}^{n} m_i \left(y_i^2 + z_i^2\right)\right]\cos^2\alpha + \left[\sum_{i=1}^{n} m_i \left(z_i^2 + x_i^2\right)\right]\cos^2\beta + \left[\sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2\right)\right]\cos^2\gamma \\ &\quad -2\left[\sum_{i=1}^{n} m_i x_i y_i\right]\cos\alpha\cos\beta - 2\left[\sum_{i=1}^{n} m_i y_i z_i\right]\cos\beta\cos\gamma - 2\left[\sum_{i=1}^{n} m_i z_i x_i\right]\cos\gamma\cos\alpha \end{aligned} \tag{4.14}$$

or by applying the definitions (4.2) and (4.4):

$$\begin{aligned} J_\Delta &= J_x \cos^2\alpha + J_y \cos^2\beta + J_z \cos^2\gamma \\ &\quad - 2J_{xy} \cos\alpha\cos\beta - 2J_{yz} \cos\beta\cos\gamma - 2J_{zx} \cos\gamma\cos\alpha \end{aligned} \tag{4.15}$$

### *4.4. Principal moments of inertia. Principal axes of inertia*

Maxima and minima of $J_\Delta$ seen as a function of three variables $\cos\alpha, \cos\beta, \cos\gamma$ which are subject to the constraint:

$$\cos^2\alpha + \cos\beta^2 + \cos\gamma^2 - 1 = 0 \tag{4.16}$$

may be found if one applies the following necessary conditions :

$$\frac{\partial\Phi}{\partial\left(\cos\alpha\right)} = 0; \quad \frac{\partial\Phi}{\partial\left(\cos\beta\right)} = 0; \quad \frac{\partial\Phi}{\partial\left(\cos\gamma\right)} = 0. \tag{4.17}$$

in which the function Φ is given by:

$$\begin{aligned} \Phi &= J_\Delta + \lambda\left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma\right) = J_x \cos^2\alpha + J_y \cos^2\beta + J_z \cos^2\gamma + \\ &-2J_{xy} \cos\alpha\cos\beta - 2J_{yx} \cos\beta\cos\gamma - 2J_{xz} \cos\gamma\cos\alpha + \lambda\left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma\right) \end{aligned} \tag{4.18}$$

The necessary conditions become:

$$\begin{aligned} &\left(J_x - \lambda\right)\cos\alpha - J_{xy} \cos\beta - J_{xz} \cos\gamma = 0 \\ &-J_{yx} \cos\alpha + \left(J_y - \lambda\right)\cos\beta - J_{yz} \cos\gamma = 0 \\ &-J_{zx} \cos\alpha + J_{zy} \cos\beta + (J_z - \lambda)\cos\gamma = 0 \end{aligned} \tag{4.19}$$

This system of three homogeneous linear equations cannot have the trivial solution $\cos\alpha = \cos\beta = \cos\gamma = 0$. It is thus necessary that the system's determinant cancels:

$$\begin{vmatrix} J_x-\lambda & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y-\lambda & J_{yz} \\ -J_{zx} & -J_{zy} & J_z-\lambda \end{vmatrix} = 0. \tag{4.20}$$

In the following, the roots of the algebraic equation obtained from developing the determinant will be denoted: $\lambda_1, \lambda_2, \lambda_3$. Two theorems concern these values:

T1. Among the roots $\lambda_1, \lambda_2, \lambda_3$ are those which represent the maximum and minimum of $J_\Delta$.
Indeed, if $\lambda_i = J_i$ is one of these roots and $\cos\alpha_i, \cos\beta_i, \cos\gamma_i$ are the corresponding values of direction cosines, then by multiplying the three equations (4.19) respectively by $\cos\alpha_i, \cos\beta_i, \cos\gamma_i$ and adding the resulting equations afterwards, it follows that:

$$\begin{aligned} &-\lambda_i\left(\cos^2\alpha_i+\cos^2\beta_i+\cos^2\gamma_i\right)+J_x\cos^2\alpha_i+J_y\cos^2\beta_i+J_z\cos^2\gamma_i \\ &-2J_{xy}\cos\alpha_i\cos\beta_i-2J_{yz}\cos\alpha_i\cos\gamma_i-2J_{zx}\cos\gamma_i\cos\alpha_i=0 \end{aligned} \tag{4.21}$$

but $\lambda_i = J_i$ (*i=1,2,3*) and because by virtue of (4.10) $\cos^2\alpha_i + \cos\beta_i^2 + \cos\gamma_i^2 - 1 = 0$, it follows:

$$\begin{aligned} J_i &= J_x\cos^2\alpha_i+J_y\cos^2\beta_i+J_z\cos^2\gamma_i \\ &\quad -2J_{xy}\cos\alpha_i\cos\beta_i-2J_{yz}\cos\alpha_i\cos\gamma_i-2J_{zx}\cos\gamma_i\cos\alpha_i. \end{aligned} \tag{4.22}$$

T2. The directions defined by the direction cosines $\cos\alpha_i, \cos\beta_i, \cos\gamma_i$ (i=1,2,3) are orthogonal.
Indeed, by multiplying the three equations (4.19) written for $\lambda = \lambda_i$, $\cos\alpha = \cos\alpha_i$, $\cos\beta = \cos\beta_i$, $\cos\gamma = \cos\gamma_i$, respectively by $\cos\alpha_j, \cos\beta_j, \cos\gamma_j$ and by adding these equations afterwards, it follows that:

$$\begin{aligned} &-\lambda_i\left(\cos\alpha_i\cos\alpha_j+\cos\beta_i\cos\beta_j+\cos\gamma_i\cos\gamma_j\right)+J_x\cos\alpha_i\cos\alpha_j \\ &+J_y\cos\beta_i\cos\beta_j+J_z\cos\gamma_i\cos\gamma_j-J_{xy}\left(\cos\alpha_i\cos\beta_j+\cos\alpha_j\cos\beta_i\right) \\ &-J_{zx}\left(\cos\gamma_i\cos\alpha_j+\cos\gamma_j\cos\alpha_i\right)-J_{yz}\left(\cos\beta_i\cos\gamma_j+\cos\beta_j\cos\gamma_i\right)=0 \end{aligned} \tag{4.23}$$

If the index *i* is replaced by *j* and at the same time the index *j* by *i*, one gets:

$$\begin{aligned} &-\lambda_j\left(\cos\alpha_j\cos\alpha_j+\cos\beta_j\cos\beta_i+\cos\gamma_j\cos\gamma_i\right)+J_x\cos\alpha_j\cos\alpha_i \\ &+J_y\cos\beta_j\cos\beta_i+J_z\cos\gamma_j\cos\gamma_i-J_{xy}\left(\cos\alpha_j\cos\beta_i+\cos\alpha_i\cos\beta_j\right) \\ &-J_{zx}\left(\cos\gamma_j\cos\alpha_i+\cos\gamma_i\cos\alpha_j\right)-J_{yz}\left(\cos\beta_j\cos\gamma_i+\cos\beta_i\cos\gamma j\right)=0 \end{aligned} \tag{4.24}$$

By subtracting equations (4.24) and (4.23), it can be obtained $\left(\lambda_i-\lambda_j\right)\left(\cos\alpha_i\cos\alpha_j+\cos\beta_i\cos\beta_j+\cos\gamma_i\cos\gamma_j\right)=0$ and because in general $\lambda_i \neq \lambda_j$ it follows that:

$$\cos\alpha_i \cos\alpha_j + \cos\beta_i \cos\beta_j + \cos\gamma_i \cos\gamma_j = 0. \tag{4.25}$$

The consequence is that these two directions are orthogonal. It also follows that:

$$\begin{aligned} J_{ij} &= J_x \cos\alpha_i \cos\alpha_j + J_y \cos\beta_i \cos\beta_j + J_z \cos\gamma_i \cos\gamma_j \\ &- J_{xy}\left(\cos\alpha_i \cos\beta_j + \cos\alpha_j \cos\beta_i\right) - J_{yz}\left(\cos\beta_i \cos\gamma_j + \cos\beta_j \cos\gamma_i\right) \\ &- J_{zx}\left(\cos\gamma_i \cos\alpha_j + \cos\gamma_j \cos\alpha_i\right) = 0 \end{aligned} \tag{4.26}$$

Therefore, the products of inertia are null in this frame. The moments of inertia $J_1, J_2, J_3$ are called **principal moments of inertia**. The axes whose direction cosines are $\cos\alpha_i, \cos\beta_i, \cos\gamma_i$ (i=1, 2, 3) are called **principal axes of inertia**. These axes are orthogonal. The products of inertia with respect to principal axes of inertia are null.

## *4.5. Moments of inertia of planar systems of material points*

If the material points are situated in the *Oxy* plane, then $z_i = 0$ and formulas (4.2) and (4.4) become:

$$\begin{aligned} &J_x = \sum_{i=1}^{n} m_i y_i^2; \quad J_y = \sum_{i=1}^{n} m_i x_i^2; \quad J_z = \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2\right) = J_x + J_y; \\ &J_{xy} = \sum_{i=1}^{n} m_i x_i y_i. \end{aligned} \tag{4.27}$$

and $J_{xz} = J_{yz} = 0$. The development of the determinant (4.20) in this case reduces to $\left[\lambda^2 - \left(J_x + J_y\right)\lambda + J_x J_y - J_{xy}^2\right]\left(\lambda - J_z\right) = 0$ and the principal moments of inertia are:

$$\begin{aligned} J_1 &= \frac{J_x + J_y}{2} + \sqrt{\left[\frac{J_x - J_y}{2}\right]^2 + J_{xy}^2} \\ J_2 &= \frac{J_x + J_y}{2} - \sqrt{\left[\frac{J_x - J_y}{2}\right]^2 + J_{xy}^2} \end{aligned} \tag{4.28}$$

The third principal moment of inertia is $J_3 = J_z = J_0$. For a plane system situated in the *Oxy* plane $\cos\beta = \sin\alpha$, $\cos\gamma = 0$ and equations (4.21) became:

$$\begin{aligned} &\left(J_x - \lambda\right)\cos\alpha - J_{xy}\sin\alpha = 0 \\ &-J_{xy}\cos\alpha + \left(J_y - \lambda\right)\sin\alpha = 0 \end{aligned} \tag{4.29}$$

For $\lambda = J_1, \alpha = \alpha_1$ and for $\lambda = J_2$, $\alpha = \alpha_2$, it follows respectively:

$$tg\alpha_1 = \frac{J_{xy}}{J_y - J_1}; \quad tg\alpha_2 = \frac{J_{xy}}{J_y - J_2} \tag{4.30}$$

### 4.6. Ellipsoid of inertia. Radius of inertia

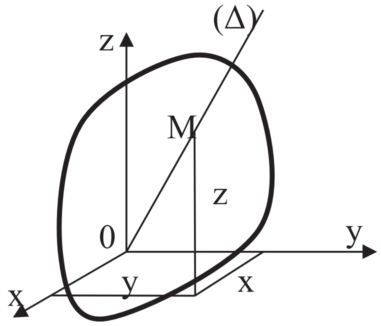


Fig. 4.4 Ellipsoid of inertia axis

Considering an axis $(\Delta)$ passing through the origin O of a Cartesian orthogonal frame Oxyz (fig. 4.4), and a point M on this axis so that

$$OM = \frac{C}{\sqrt{J_\Delta}}, \tag{4.31}$$

where C is an arbitrary constant, then the coordinates of the point M are:

$$x = \frac{C}{\sqrt{J_\Delta}}\cos\alpha; \qquad y = \frac{C}{\sqrt{J_\Delta}}\cos\beta; \qquad z = \frac{C}{\sqrt{J_\Delta}}\cos\gamma. \tag{4.32}$$

If $\cos\alpha, \cos\beta, \cos\gamma$ are substituted from (4.32) into (4.15), it follows that:

$$J_x x^2 + J_y y^2 + J_z z^2 - 2J_{xy}xy - 2J_{yz}yz - 2J_{zx}zx = C^2. \tag{4.33}$$

This represents the equation of an ellipsoid, called the **ellipsoid of inertia**. If *Ox*, *Oy* and *Oz* are the principal axes of inertia then $J_x = J_1; J_y = J_2; J_z = J_3;$ $J_{xy} = 0; J_{yz} = 0; J_{zx} = 0$ and (4.33) becomes:

$$J_1 x^2 + J_2 y^2 + J_3 z^2 = C^2. \tag{4.34}$$

If the material points are situated in the 0xy plane, then this equation becomes

$$J_1 x^2 + J_2 y^2 = C^2. \tag{4.35}$$

This is the equation of an ellipse, called the **ellipse of inertia**. By definition

$$i_\Delta = \sqrt{\frac{J_\Delta}{M}}, \tag{4.36}$$

represents the **radius of inertia** or **radius of gyration**.

### 4.7. Moments of inertia of some material lines, material surfaces and material bodies

As applications of the given definitions, in this paragraph are determined the principal moments of inertia for some bodies, with possible technical applications.

#### 4.7.1. The rod

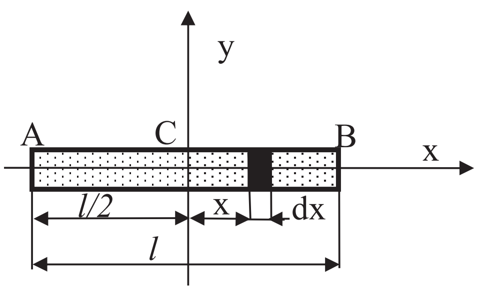


Fig. 4.5 The homogeneous rod

It is considered a homogeneous rod of length *l* and of mass *M* (Fig. 4.5) which represents the simplest one dimensional example.

The moment of inertia with respect to the axis Oz , since the elementary mass can be obtained as *dm=M/l· dx*, results as:

$$J_z = \int x^2 dm = \int_{-l/2}^{l/2} x^2 \frac{M}{l} dx = \frac{Mx^3}{3l}\bigg|_{-l/2}^{l/2} = \frac{Ml^2}{12}. \quad (4.37)$$

If the Oz axis is normal in A on the rod, the moment of inertia is:

$$J_z = \int x^2 dm = \int_0^l x^2 \frac{M}{l} dx = \frac{Mx^3}{3l}\bigg|_0^l = \frac{Ml^2}{3}. \quad (4.38)$$

### 4.7.2. Circular disc

A homogeneous circular disc of radius *R*, mass *M* (Fig. 4.6) and arbitrary constant thickness is considered as a two dimensional example.

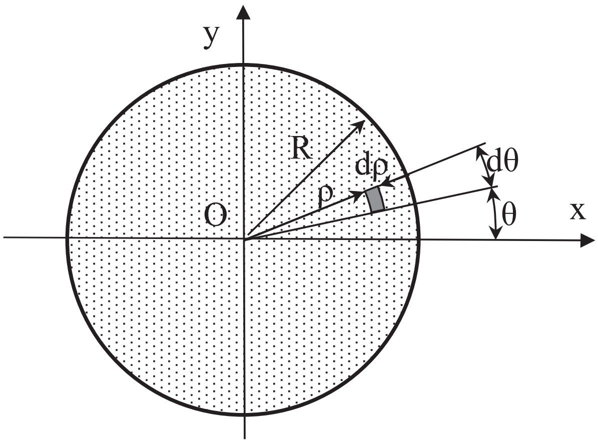


**Fig. 4.6 Circular disc**

The moment of inertia with respect to the Oz axis can be determined using as elementary mass $dm = \frac{M}{\pi R^2} dA = \frac{M}{\pi R^2} \rho \, d\rho \, d\theta$:

$$J_z = \int \rho^2 dm = \int\int \rho^2 \frac{M}{\pi R^2} \rho d\rho d\theta = \frac{M}{\pi R^2} \int_0^{2\pi} d\theta \int_0^R \rho^3 d\rho = \frac{M}{\pi R^2} \theta \bigg|_0^{2\pi} \frac{\rho^4}{4}\bigg|_0^R = \frac{MR^2}{2} \quad (4.39)$$

The moment of inertia about the mass center O is for symmetry reasons $J_O = \frac{1}{2}\left(J_x + J_y + J_z\right) = J_z$, from which

$$J_x = J_y = \frac{J_z}{2} = \frac{MR^2}{4}. \quad (4.40)$$

### 4.7.3. Rectangular plate

A homogeneous rectangular plate (Fig. 4.7) of dimensions *a, b* and of mass *M* is considered in this case. The moments of inertia with respect to the *Ox, Oy* and *Oz* axes, using as the elementary mass $dm = \frac{M}{ab} dA = \frac{M}{ab} dxdy$ are:

$$J_x = \int y^2 dm = \int\int y^2 \frac{M}{ab} dxdy = \frac{M}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} y^2 dy = \frac{M}{ab} x\bigg|_{-a/2}^{a/2} \cdot \frac{y^3}{3}\bigg|_{-b/2}^{b/2} = \frac{Mb^2}{12} \quad (4.41)$$

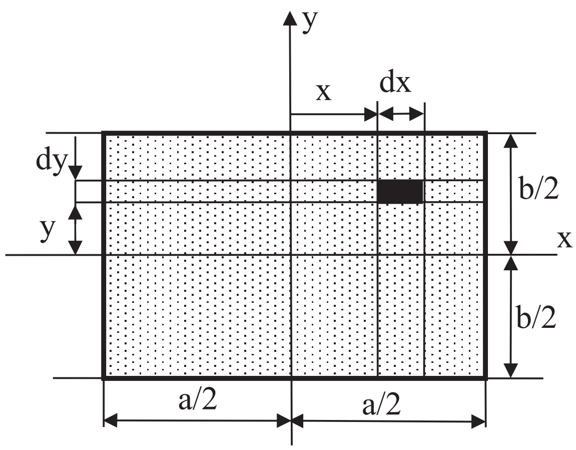


**Fig. 4.7 Rectangular plate**

and respectively:

$$J_y = \int x^2 dm = \frac{Ma^2}{12}; \qquad J_z = \int \left(x^2 + y^2\right) dm = J_x + J_y = \frac{M\left(a^2 + b^2\right)}{12} \qquad (4.42)$$

In the particular case *b=0*, it can be found the case of the rod (4.7.1).

### 4.7.4. Rectangular parallelepiped

Consider a homogeneous rectangular parallelepiped (Fig. 4.8) of dimensions *a, b, c* and mass *M*. The moments of inertia with respect to the central principal axes *Ox, Oy* and *Oz*, using as the elementary mass $dm = \frac{M}{abc} dV = \frac{M}{abc} dxdydz$ are

$$J_x = \int \left(y^2 + z^2\right) dm = \iiint \left(y^2 + z^2\right) \frac{M}{abc} dxdydz$$
$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \left[ \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} \left(y^2 + z^2\right) dzdy \right] = \frac{M\left(b^2 + c^2\right)}{12} \qquad (4.43)$$

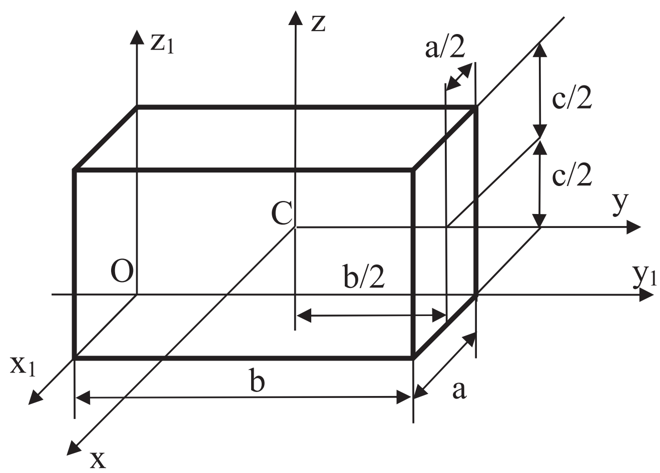


**Fig. 4.8 Rectangular parallelepiped**

The other two moments of inertia, can be deduced in a similar manner:

$$J_y = \frac{M\left(c^2 + a^2\right)}{12}; \qquad J_z = \frac{M\left(a^2 + b^2\right)}{12} \tag{4.44}$$

If $c=0$ in these equations, the particular case of the rectangular plate (4.7.3) is recovered. If $b=c=0$, the particular case of the rod (4.7.1) of length $l=a$ will be obtained.

### 4.7.5. Circular cylinder

Consider a homogeneous cylinder of revolution of radius *R*, height *H* and of mass *M*. The moment of inertia with respect to the *Oz* axis, using as the elementary mass $dm = \frac{M}{\pi R^2 H} dV = \frac{M}{\pi R^2 H} \rho\, d\rho\, d\theta\, dz$ is:

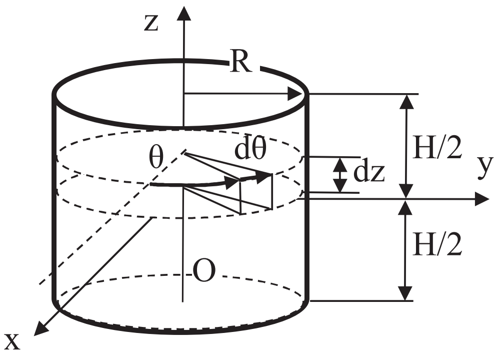


**Fig. 4.9 Circular cylinder**

$$\begin{aligned} J_z &= \int\left(x^2 + y^2\right)dm = \int \rho^2 dm = \iiint \rho^2 \frac{M}{\pi R^2 H} \rho\, d\rho\, d\theta\, dz \\ &= \frac{M}{\pi R^2 H} \int_{-H/2}^{H/2} dz \int_0^{2\pi} d\theta \int_0^R \rho^3 d\rho = \frac{MR^2}{2} \end{aligned} \tag{4.45}$$

The moment of inertia about the mass center O is:

$$\begin{aligned} J_O &= \int\left(x^2 + y^2 + z^2\right)dm = \int\left(\rho^2 + z^2\right)dm = \frac{MR^2}{2} + \iiint z^2 \frac{M}{\pi R^2 H} \rho\, d\rho\, d\theta\, dz \\ &= \frac{MR^2}{2} + \frac{M}{\pi R^2 H} \int_{-H/2}^{H/2} z^2 dz \int_0^{2\pi} d\theta \int_0^R \rho\, d\rho = \frac{MR^2}{2} + \frac{M}{\pi R^2 H} \frac{H^3}{12} 2\pi \frac{R^2}{2} \\ &= \frac{MR^2}{2} + \frac{MH^2}{12} = \frac{1}{2}\left(J_x + J_y + J_z\right) \end{aligned} \tag{4.46}$$

From the last two equations, due to the symmetry about the *Oz* axis, it follows:

$$J_x = J_y = M\left(\frac{R^2}{4} + \frac{H^2}{12}\right). \tag{4.47}$$

For $R=0$ can be obtained the case of the rod (4.7.1) and for $H=0$ will be obtained the case of a disc (4.7.2).

### 4.7.6. The circular cone

Consider a homogeneous cone of height *h*, having the radius of the circle of the base *R* and mass *M*.

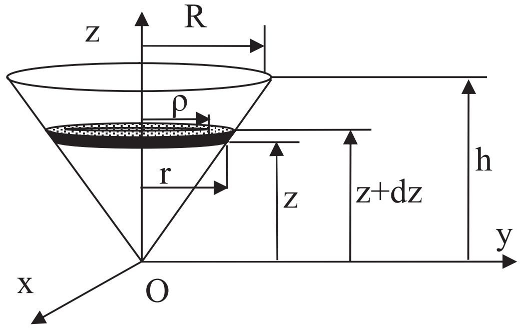


**Fig. 4.10 Circular cone**

The elementary mass is (Fig. 4.10) is

$$dm = \frac{M}{\pi R^2 h/3} dV = \frac{3M}{\pi R^2 h} 2\pi\rho d\rho dz = \frac{6M}{R^2 h}\rho d\rho dz \tag{4.48}$$

and the moment of inertia about the Oz axis is:

$$\begin{aligned} J_z &= \int \left(x^2 + y^2\right) dm = \frac{6M}{R^2 h}\iint \rho^3 d\rho dz = \frac{6M}{R^2 h}\int_0^h \left( \int_0^r \rho^3 d\rho \right) dz = \frac{6M}{R^2 h}\int_0^h \frac{r^4}{4} dz \\ &= \frac{6M}{R^2 h}\int_0^h \frac{R^4 z^4}{4h^4} dz = \frac{3MR^2}{2h^5}\frac{h^5}{5} = \frac{3}{10}MR^2 \end{aligned} \tag{4.49}$$

### 4.7.7. Sphere

A homogeneous sphere of radius *R* and of mass *M* is considered. The elementary mass is determined as the mass of the volume situated between a sphere of radius $\rho$ and a sphere of a radius $\rho + d\rho$, it follows that:

$$dm = \frac{M}{4\pi R^3/3} dV = \frac{M}{4\pi R^3/3} 4\pi\rho^2 d\rho = \frac{3M}{R^3}\rho^2 d\rho \tag{4.50}$$

The moment of inertia of the sphere with respect to its center is:

$$J_o = \int \left(x^2 + y^2 + z^2\right) dm = \int \rho^2 dm = \int_0^R \rho^2 \frac{3M}{R^3}\rho^2 d\rho = \frac{3M}{R^3}\frac{\rho^5}{5}\Bigg|_0^R = \frac{3}{5}MR^2 \tag{4.51}$$

The following expressions are also useful in applications:

$$\begin{aligned} &J_x = J_y = J_z;\; J_x + J_y + J_z = 2J_O \Rightarrow 3J_x = 2J_O \\ &\Rightarrow J_x = J_y = J_z = \frac{2}{3}J_O = \frac{2}{5}MR^2 \end{aligned} \tag{4.52}$$

<u>Example</u>. Determine $J_x$, $J_y$, $J_{xy}$, the principal moments of inertia, and determine the principal axes of inertia of the plate shown in the next figure.

This plate can be divided in two rectangular plates whose areas and masses are:

$$A_1 = 2a \cdot 6a = 12a^2; \; A_2 = 2a \cdot 2a = 4a^2; \; A = A_1 + A_2 = 16a^2;$$

$$M_1 = \frac{M}{16a^2}12a^2 = \frac{3}{4}M; \; M_2 = \frac{M}{16a^2}4a^2 = \frac{M}{4}.$$

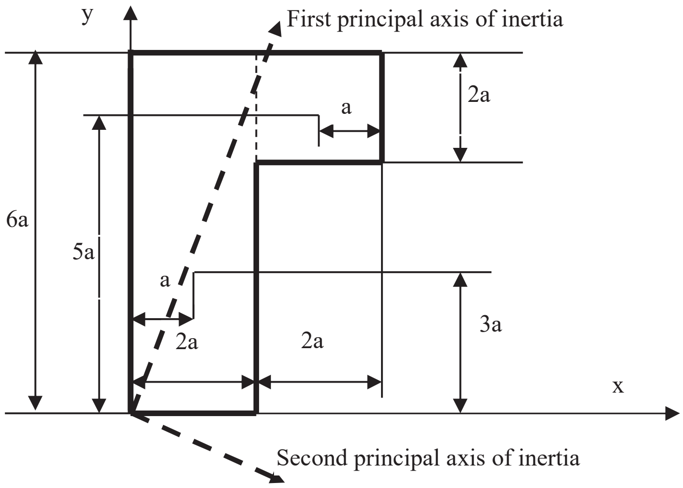


**Fig. 4.11 Geometry of the plate**

The moments of inertia $J_x$, $J_y$ and the product of inertia $J_{xy}$ are:

$$J_x = \frac{3M}{4}\frac{(6a)^2}{12} + \frac{3M}{4}(3a)^2 + \frac{M}{4}\frac{(2a)^2}{12} + \frac{M}{4}(5a)^2 = \frac{46Ma^2}{3}$$

$$J_y = \frac{3M}{4}\frac{(2a)^2}{12} + \frac{3M}{4}(a)^2 + \frac{M}{4}\frac{(2a)^2}{12} + \frac{M}{4}(3a)^2 = \frac{10Ma^2}{3}$$

$$J_{xy} = 0 + \frac{3Ma}{4}3a + 0 + \frac{M3a}{4}5a = 6Ma^2$$

The principal moments of inertia are:

$$J_{1,2} = \frac{\frac{46Ma^2}{3} + \frac{10Ma^2}{3}}{2} \pm \sqrt{\left(\frac{\frac{46Ma^2}{3} - \frac{10Ma^2}{3}}{2}\right)^2 + \left(6Ma^2\right)^2}$$

It follows that $J_1 \approx 17.80Ma^2$; $J_2 \approx 0.75Ma^2$.

The angles $\alpha_1$ and $\alpha_2$ corresponding to the principal axes of inertia are:

$$\tan\alpha_1 = \frac{J_{xy}}{J_y - J_1} = \frac{6Ma^2}{\frac{10Ma^2}{3} - 17.8Ma^2} \cong -0.414 \Rightarrow \alpha_1 \cong -22^\circ 30'$$

$$\tan\alpha_2 = \frac{J_{xy}}{J_y - J_2} = \frac{6Ma^2}{\frac{10Ma^2}{3} - 0.75Ma^2} \cong 2.414 \Rightarrow \alpha_2 \cong 67^\circ 30'$$

# 5. STATICS OF A MATERIAL POINT

## 5.1. The free material point. The constrained material point

A free material point can occupy any position in space under the action of a certain system of forces. A material point or particle whose ability to move is restricted by constraints is a called a **constrained material point**. These restrictions are usually expressed in such a way that the coordinates of the material point should satisfy certain equations or inequalities during the motion. A constraint is **bilateral** when the restriction is expressed by equations linking the co-ordinates of the material point. A constraint is **unilateral** when the restriction is expressed by inequalities.

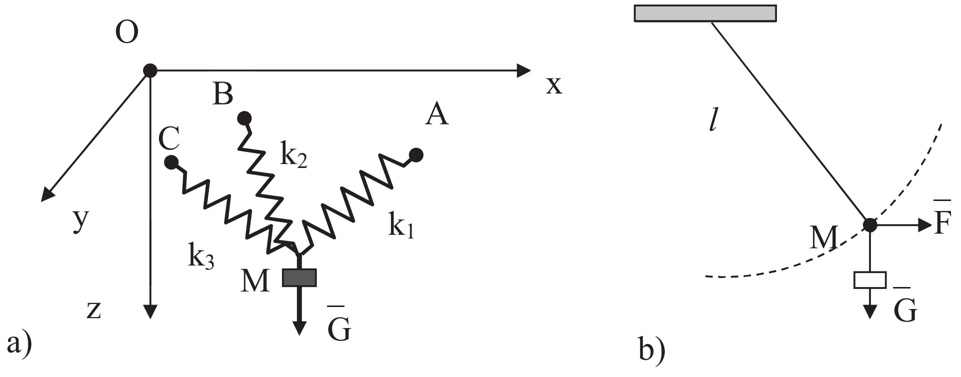


**Fig. 5.1 Free (a) and constrained (b) material point**

For example the material point shown in Fig. 5.1a is a free one, because the three springs do not restrict its ability to move, while the material point shown in Fig. 5.1b is a constrained one. More precisely, if *OM=l* is a rod the constraint is bilateral because it is expressed by equation $x^2 + y^2 + z^2 = l^2$ and if OM is an inextensible string, the constraint is unilateral, because it is expressed by the inequality $x^2 + y^2 + z^2 \leq l^2$, the point could eventually reach positions inside the sphere of radius *l*.

## 5.2. Statics of free material point

### 5.2.1. Concurrent forces

The forces acting on the same free material point evidently have the same point of application. They form a concurrent system of forces. By virtue of the principle of the parallelogram, these forces may be in general replaced by a unique force, the resultant force, using the polygon of forces. It can be written:

$$\overline{R} = \overline{F}_1 + \overline{F}_2 + ...\overline{F}_n = \sum_{i=1}^{n} \overline{F}_i \qquad (5.1)$$

If the projections of a certain force $\bar{F}_i$, on the Ox, Oy, Oz axes are

$$X_i = |\bar{F}_i|\cos\alpha_i;\ Y_i = |\bar{F}_i|\cos\beta_i;\ Z_i = |\bar{F}_i|\cos\gamma_i; \tag{5.2}$$

and if (5.1) is projected on the same axes, is follows that:

$$X = \sum_{i=1}^{n} X_i;\ \ Y = \sum_{i=1}^{n} Y_i;\ \ Z = \sum_{i=1}^{n} Z_i; \tag{5.3}$$

From (5.3) we can deduce the magnitude (modulus) $|\bar{R}|$ of the sum vector $\bar{R}$ and the values of **direction cosines** cosα, cosβ, cosγ as:

$$|\bar{R}| = \sqrt{X^2 + Y^2 + Z^2}$$

$$\cos\alpha = \frac{X}{\sqrt{X^2 + Y^2 + Z^2}};\ \cos\beta = \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}};\ \cos\gamma = \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}; \tag{5.4}$$

### 5.2.2. Equilibrium of a system of concurrent forces

When $\bar{R} = 0$, the system of concurrent forces is in a **state of equilibrium**. If a free material point is at rest and a system of concurrent force in a state of equilibrium is acting on it, then the material point continues to remain at rest, by virtue of the principle of inertia. The necessary and sufficient condition for a system of concurrent forces to be in a state of equilibrium is $\bar{R} = \bar{0}$ or see (5.4):

$$\sum_{i=1}^{n} X_i = 0;\ \ \sum_{i=1}^{n} Y_i = 0;\ \ \sum_{i=1}^{n} Z_i = 0 \tag{5.5}$$

Example. Consider a free point M(Fig. 5.1a). If A($x_1$, $y_1$,$z_1$), B($x_2$, $y_2$, $z_2$), C($x_3$, $y_3$, $z_3$), are fixed hanging points to which M(x, y, z) is connected through three springs MA, MB, MC of elastic constants $k_1$, $k_2$, $k_3$. G is the weight attached to the point M, determine x, y, z for the state in which M is at rest.

The forces acting on the material point are:

$$\bar{F}_1 = k_1\overline{MA};\ \ \bar{F}_2 = k_2\overline{MB};\ \ \bar{F}_3 = k_3\overline{MC}$$

It follows that

$$\bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{G} = \bar{0} \text{ or } k_1\overline{MA} + k_2\overline{MB} + k_3\overline{MC} + \bar{G} = \bar{0}$$

The projections of this equation on Ox, Oy and Oz axes are:

$$k_1(x_1 - x) + k_2(x_2 - x) + k_3(x_3 - x) = 0$$

$$k_1(y_1 - y) + k_2(y_2 - y) + k_3(y_3 - y) = 0$$

$$k_1(z_1 - z) + k_2(z_2 - z) + k_3(z_3 - z) - G = 0$$

It follows that:

$$x = \frac{k_1x_1 + k_2x\ + k_3x_3}{k_1 + k_2 + k_3};\ \ y = \frac{k_1y_1 + k_2y_2 + k_3y_3}{k_1 + k_2 + k_3};\ \ z = \frac{k_1z_1 + k_2z_2 + k_3z_3 - G}{k_1 + k_2 + k_3}.$$

## *5.3. Statics of constrained material points*

### 5.3.1. The axiom of constraints

The principles established by Newton are valid only for a free material point. If for example a material point is constrained to move on a circle situated in a vertical plane, it can be at rest at a position A (Fig. 5.2a).

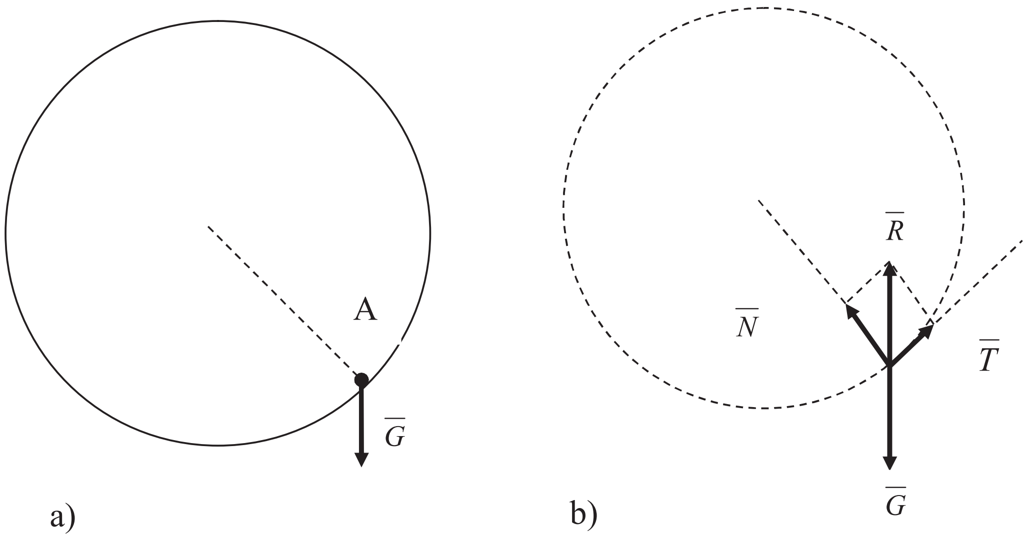


**Fig. 5.2 A constrained material point (a) and the equivalent forces of the constraint (b)**

The force $\overline{F} = \overline{G}$ acting on this material point is not null. It follows that the principle of inertia is not applicable in this case. If it is assumed a free material point A(Fig. 5.2b) under the action of two forces $\overline{F}$ and $\overline{R} = -\overline{F}$, it will be also at rest (the circle is shown by a broken line). It can be concluded that the situations in Fig. 5.2a and Fig. 5.2b are physically equivalent. The force $\overline{R}$ is called **reaction**.

The component $\overline{N}$ of the reaction, perpendicular to the tangent, is called the **normal reaction**, the tangential component $\overline{T}$ is called **tangential reaction** or **friction**.

Similarly, if a material point is constrained to remain on a certain surface, then the vector component of the reaction perpendicular to the surface is called the normal reaction, whereas the tangential vector component is called the tangential reaction or friction.

Therefore, whenever it is assumed that there is no friction, it is implicitly assumed that the reaction is perpendicular to the curves or surfaces, which are in this case called **smooth constraints**.

This replacement of physical constraints by reaction forces (and by moments in some cases) is allowed by the axiom (principle) of constraints.

### 5.3.2. Smooth constraint produced by a surface or a curve

In the first case, the reaction $\overline{R} = \overline{N}$. If the material point A is constrained to remain on a surface f( x, y, z)=0, then the reaction $\overline{N} = \lambda \cdot grad(f)$ and the condition that the material point is at rest $\overline{F} + \overline{N} = \overline{0}$, becomes $\overline{F} + \lambda \cdot grad(f) = \overline{0}$ or

$$X + \lambda \frac{\partial f}{\partial x} = 0;\ Y + \lambda \frac{\partial f}{\partial x} = 0;\ Z + \lambda \frac{\partial f}{\partial x} = 0, \tag{5.6}$$

in which λ is a scalar which remains to be determined.

Note. The gradient of a function of three variables is by definition $grad\left[f(x,y,z)\right] = \frac{\partial f}{\partial x}\overline{i} + \frac{\partial f}{\partial y}\overline{j} + \frac{\partial f}{\partial z}\overline{k}$.

If the material point A is constrained to remain on a curve defined as intersection of two surfaces $\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$ then the reaction can be written as a sum of two components $\overline{N} = \lambda \cdot grad(f) + \mu \cdot grad(g)$, in which λ and μ are two scalars to be determined. The condition for the material point A to be at rest $\overline{F} + \overline{N} = \overline{0}$ becomes $\overline{F} + \lambda \cdot grad(f) + \mu \cdot grad(g) = \overline{0}$ or the system of scalar equations:

$$X + \lambda \frac{\partial f}{\partial x} + \mu \frac{\partial g}{\partial x} = 0;\quad Y + \lambda \frac{\partial f}{\partial y} + \mu \frac{\partial g}{\partial y} = 0;\quad Z + \lambda \frac{\partial f}{\partial z} + \mu \frac{\partial g}{\partial z} = 0. \tag{5.7}$$

Equations (5.6) and (5.7) make it possible to find the coordinates of point A at equilibrium and the vector components of the reaction $\overline{N}$. If only the coordinates of A are required, then it is necessary to eliminate λ and μ. For a smooth surface the following system of equations is to be solved:

$$\begin{cases} f(x,y,z) = 0 \\ \dfrac{x}{\frac{\partial f}{\partial x}} = \dfrac{y}{\frac{\partial f}{\partial y}} = \dfrac{z}{\frac{\partial f}{\partial z}} \end{cases} \tag{5.8}$$

and for the case of a smooth constraining curve, the system to be solved is:

$$f(x,y,z) = 0;\quad g(x,y,z) = 0;\quad \begin{vmatrix} X & Y & Z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = 0 \tag{5.9}$$

Example 1.

A material point A is constrained to move on the sphere $x^2+y^2+z^2-R^2=0$. If $A_0(x_0,y_0,z_0)$ is a fixed point and an elastic spring $A_0A$ of constant characteristic $k$ is connecting $A_0$ and A, determine the coordinates of the equilibrium positions of A.

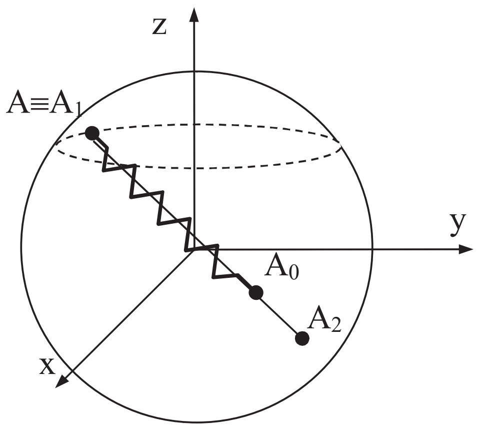


**Fig. 5.3 Material point on a smooth sphere**

Answer.

The points on the sphere verify $f(x,y,z)=0$ with $f\left(x,y,z\right)=x^2+y^2+z^2-R^2$. The computations are successively:

$$\frac{\partial f}{\partial x}=2x;\quad \frac{\partial f}{\partial y}=2y;\quad \frac{\partial f}{\partial z}=2z;$$

$$X=k(x_0-x);\quad Y=k(y_0-y);\quad Z=k(z_0-z);$$

$$x^2+y^2+z^2-R^2=0;$$

$$\frac{k(x_0-x)}{2x}=\frac{k(y_0-y)}{2y}=\frac{k(z_0-z)}{2z}$$

Two positions of rest are obtained:

$$A_1\left(\frac{x_0R}{\sqrt{x_0^2+y_0^2+z_0^2}},\frac{y_0R}{\sqrt{x_0^2+y_0^2+z_0^2}},\frac{z_0R}{\sqrt{x_0^2+y_0^2+z_0^2}}\right)$$

$$A_2\left(-\frac{x_0R}{\sqrt{x_0^2+y_0^2+z_0^2}},-\frac{y_0R}{\sqrt{x_0^2+y_0^2+z_0^2}},-\frac{z_0R}{\sqrt{x_0^2+y_0^2+z_0^2}}\right)$$

Note

1. The segment $A_1A_2$ is the diameter of the sphere passing through $A_0$;

2. If $A_0$ coincides with the center O of the sphere, the material point A is in a state of rest at any point of the sphere.

Example 2.

Determine the force $\overline{F}$ acting on the material point A situated on a inclined plane (Fig. 5.4).

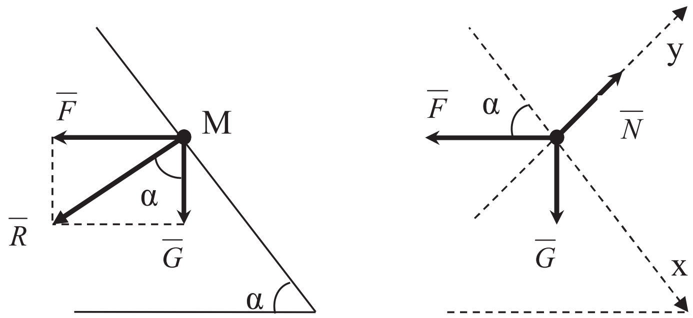


**Fig. 5.4 Material point on an inclined plane**

The condition of equilibrium $\overline{F}+\overline{G}+\overline{N}=\overline{0}$ can be written:

$$\left(\sum X_i = 0\right) \quad F\cos\alpha - G\sin\alpha = 0$$

$$\left(\sum Y_i = 0\right); \quad N - F\sin\alpha - G\cos\alpha = 0$$

If follows that:

$$F = G\tan\alpha\,; \quad N = G \quad \cos\alpha\,.$$

### 5.3.3. Rough constraints produced by a surface or a curve

In this case the reaction force is made of two forces $\overline{R}=\overline{N}+\overline{T}$. The vector component $\overline{T}$ is opposite to the tendency of sliding (Fig. 5.5a) and is called **friction force.** By experiments, Coulomb showed that

$$\left|\overline{T}\right| \le \mu\left|\overline{N}\right| \tag{5.10}$$

where $\mu$ is by definition the **coefficient of friction**.

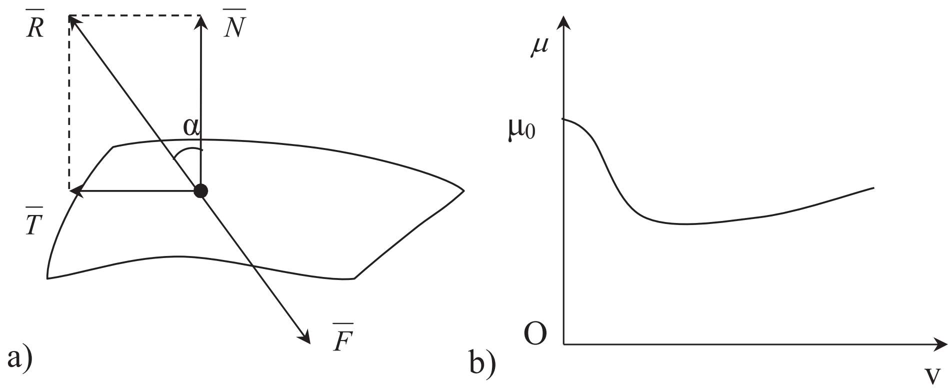


**Fig. 5.5 Material point on a rough surface (a). Sliding friction coefficient as function of velocity (b)**

There are in fact a coefficient of static friction $\mu_0$ and a coefficient of kinetic friction $\mu$. The static friction is the friction produced between the material point and the curve or surface at relative rest and the kinetic friction is the friction occurring in the relative motion between the material point and the curve or surface. The coefficient of static friction is larger than the coefficient of kinetic friction (Fig. 5.5b). Theoretically this coefficient is any positive real number and experiments with values of $\mu_0$>1 can be easily imagined. However for most practical applications this coefficient is sensibly less than one.
By definition the angle:

$$\varphi = arctg \quad \mu \tag{5.11}$$

is called **angle of friction**. The significance of φ is the following: let α be the angle of the reaction R to the normal to the surface (or to the normal plan to the curve). Then φ is the largest possible angle α.

If the material point is constrained to move on the surface f(x,y,z)=0, the angle α between the reaction $\bar{R}$ and the normal to the surface, the same as the angle between force $\bar{F}$ and the gradient of *f(x,y,z),* results from the scalar product $\bar{F} \cdot \mathrm{grad}\, f = \left|\bar{F}\right| \cdot \left|\mathrm{grad}\, f\right| \cdot \cos\alpha$ and because $\alpha \le \varphi$, then

$\cos\alpha \ge \cos\varphi = \dfrac{1}{\sqrt{1+\mathrm{tg}^2\varphi}} = \dfrac{1}{\sqrt{1+\mu^2}}$ and it follows that

$$\frac{\left|X\dfrac{\partial f}{\partial x} + Y\dfrac{\partial f}{\partial y} + Z\dfrac{\partial f}{\partial z}\right|}{\sqrt{Z^2+Y^2+Z^2}\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2}} \ge \frac{1}{\sqrt{1+\mu^2}} \tag{5.12}$$

If the material point is constrained to move on the curve defined in the parametric form as $x = x(\lambda),\ y = y(\lambda),\ z = z(\lambda)$, the angle $\beta = 90^\circ - \alpha$ between the reaction $\bar{R}$ and the tangent to the curve can be easily determined. This $\beta$ angle is also made between the force $\bar{F}$ and the tangent unit vector (which is defined as $\bar{\tau} = \dfrac{d\bar{r}}{d\lambda}$). The scalar product $\bar{F}\bar{\tau} = \left|\bar{F}\right| \cdot \left|\bar{\tau}\right| \cos\beta$ allows getting the angle $\beta$. Since $\alpha \le \varphi$, then $\dfrac{\pi}{2} - \beta \le \varphi$, or $\beta \ge \dfrac{\pi}{2} - \varphi$, which leads to

$$\cos\beta \le \cos(\pi/2 - \varphi) \Rightarrow \cos\beta \le \sin\varphi = \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}}.$$

It follows that for equilibrium on a rough curve, the following condition must be met:

$$\frac{\left| X\frac{\partial x}{\partial \lambda} + Y\frac{\partial y}{\partial \lambda} + Z\frac{\partial z}{\partial \lambda} \right|}{\sqrt{Z^2 + Y^2 + Z^2}\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2}} \leq \frac{\mu}{\sqrt{1+\mu^2}} \tag{5.13}$$

By definition the cone whose apex coincides with the material point and which is delimited by the support lines of the reaction forces at their maximum inclination relative to the normal direction to the surface (or respectively to the normal plane to the curve) is called the **cone of friction**. The axis of the cone of friction is the normal to the surface (or respectively the local tangent to the curve). The angle between the generator and the axis of the cone is φ for a surface and π/2-φ for a curve (Fig. 5.6).

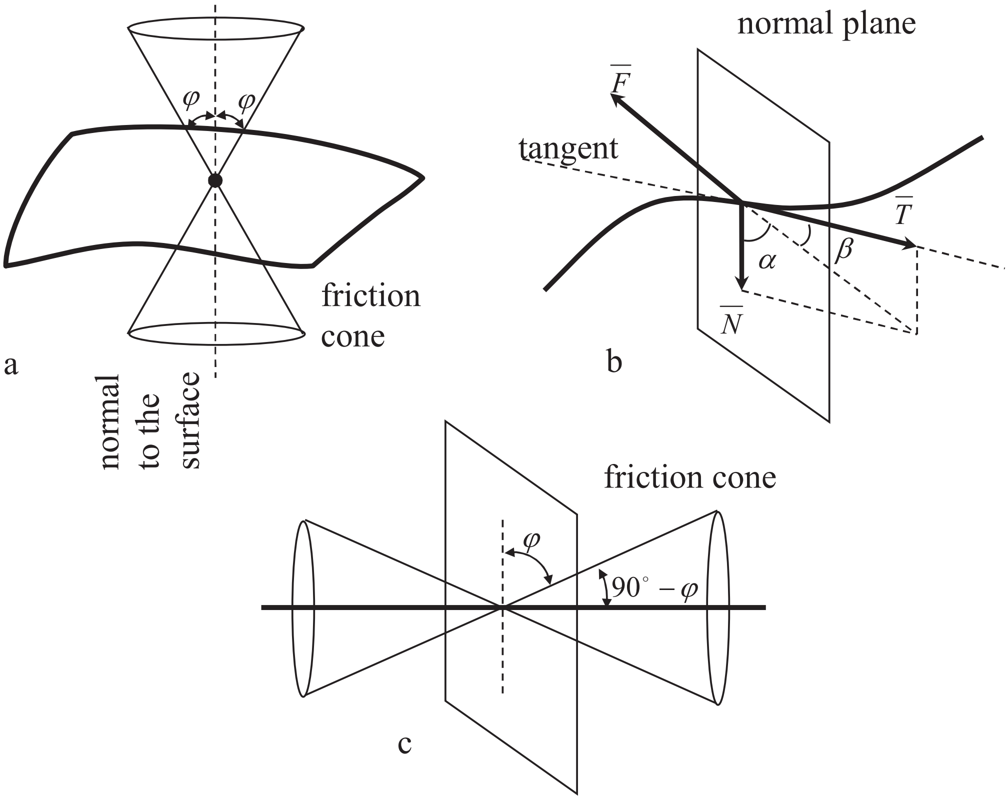


**Fig. 5.6 Cone of friction for a surface (a) and for a curve (b,c)**

The following rules can be deduced. The material point is at rest on a rough surface if the line of action of the applied force $\overline{F}$ is situated inside the cone of friction. It is at rest on a rough curve if the line of action of $\overline{F}$ is situated outside the cone of friction.

<u>Example 1</u>.
Determine the positions at rest of a material point constrained to move on the rough sphere $x^2+y^2+z^2-R^2=0$, if the coefficient of friction is $\mu$ and the weight of the material point is $G$.

The following equations can be written:

$$X = 0,\ \ Y = 0,\ \ Z = -G;\ \ f(x,y,z) = x^2 + y^2 + z^2 - R^2$$

$$\frac{\partial f}{\partial x} = 2x;\ \ \frac{\partial f}{\partial y} = 2y;\ \ \frac{\partial f}{\partial z} = 2z$$

$$\frac{\left|-G\ 2z\right|}{G\sqrt{4x^2 + 4y^2 + 4z^2}} \geq \frac{1}{\sqrt{1+\mu^2}} \qquad |z| \geq \frac{R}{\sqrt{1+\mu^2}}$$

If $z>0$, $z \geq \frac{R}{\sqrt{1+\mu^2}}$; if $z<0$, $z \leq -\frac{R}{\sqrt{1+\mu^2}}$. The material point is at rest on two zones of the sphere, one is up on the exterior of the sphere and the other is down on the inside (Fig. 5.7).

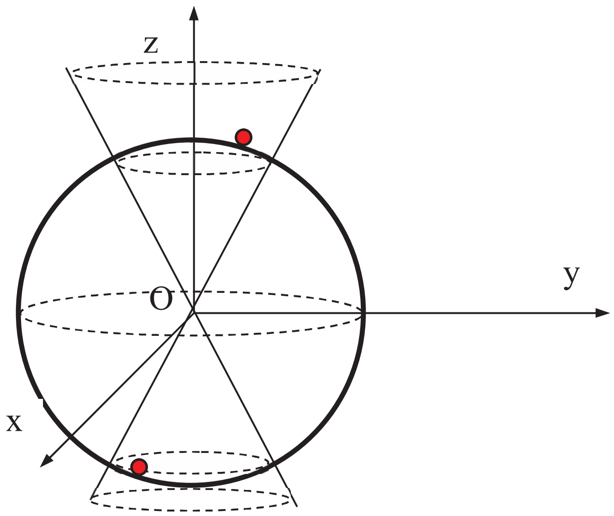


**Fig. 5.7 Equilibrium of a material point on a rough sphere**

The equilibrium zones can also be obtained as intersections of the sphere with the cone of friction as shown on the same figure.

Example 2.

Determine the positions at rest of a material point constrained to move on the rough helix: $x = R\ \cos\theta,\ \ y = R\ \sin\theta,\ \ z = (h/2\pi)\theta$, if the coefficient of friction is $\mu$ and the weight of the material point is $\overline{G}$.

The following equations can be written, taking $\theta$ as parameter in place of $\lambda$:

$X = 0, Y = 0, Z = -G$, $x' = -R\sin\theta,\ \ y' = R\cos\theta,\ \ z' = h/2\pi$.

$$\frac{\left|-G\frac{h}{2\pi}\right|}{G\sqrt{(-R\sin\theta)^2 + R^2\cos^2\theta + \frac{h^2}{4\pi^2}}} \leq \frac{\mu}{\sqrt{1+\mu^2}}$$

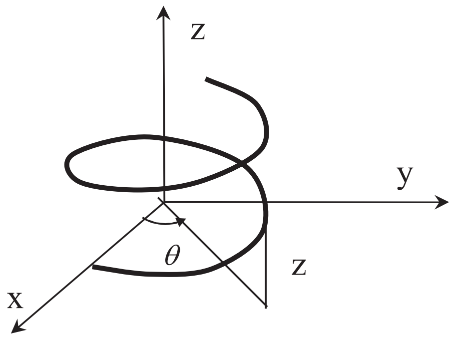


**Fig. 5.8 Equilibrium on a helix**

$$\frac{h}{\sqrt{h^2 + 4\pi^2 R^2}} \le \frac{\mu}{\sqrt{1+\mu^2}}\,; \qquad \mu \ge \frac{2\pi R}{h}.$$

If this inequality is verified, the material point remains at rest if placed in any point of the helix. If it is not verified, then no position of equilibrium exists for the material point on the helix.

# 6. SYSTEMS OF SLIDING VECTORS

## 6.1. The property of sliding vector for a force acting on a rigid body

A body which, despite the action of forces, does not undertake any deformations (i.e. in which the mutual distances of the points of the body do not change) is a **rigid body**. Particularly, two forces acting in A and B, equal in modulus but opposite in sense and having the line of action AB, have no effect on the rigid body (Fig. 6.1a)

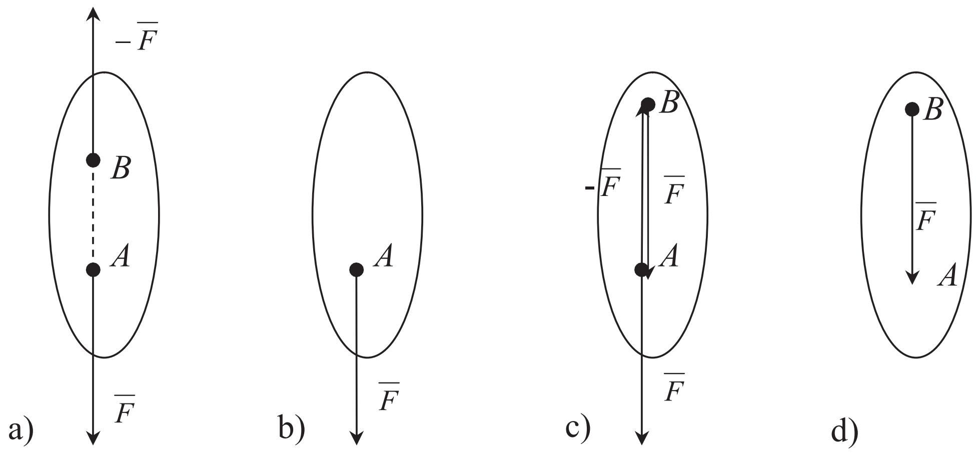


**Fig. 6.1 Forces as sliding vectors acting on a rigid body**

Consider a force $\overline{F}$ acting in *A* on the rigid body (Fig. 6.1b). Two forces are introduced: $-\overline{F}$ in *A* and $\overline{F}$ in *B* (Fig. 6.1c). The opposite forces $\overline{F}$ and $-\overline{F}$ acting in *A* (Fig. 6.1d) can be eliminated. The system of forces shown in Fig. 6.1b, and Fig. 6.1c are evidently equivalent. Therefore, if *A*, the point of application of a force $\overline{F}$, is displaced in *B* on the line of action of the force, its effect on the rigid body is the same. The property of being **sliding vectors** for forces acting on a rigid body is a direct consequence of these operations.

## 6.2. The moment of a force with respect to a point

By definition, the moment of a force with respect to a point O is a vector equal to the vector product of the radius vector $\overline{r}$ of the point of application of the force and force $\overline{F}$ (fig. 6.2) i.e.

$$\overline{M}_0\left(\overline{F}\right) = \overline{r} \times \overline{F} . \tag{6.1}$$

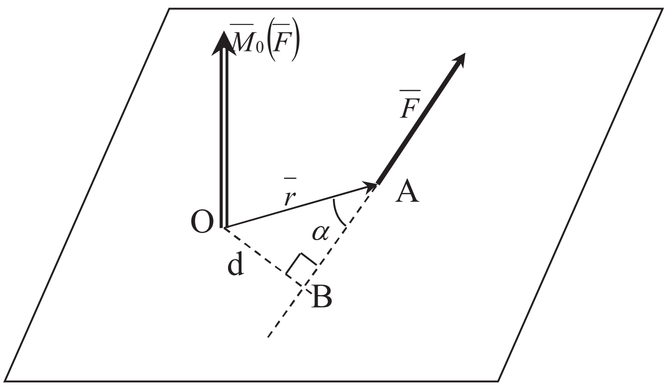


**Fig. 6.2 The geometric configuration for the moment of a force**

It follows that $\overline{M}_0\left(\overline{F}\right)$ is perpendicular to the plane determined by the point O and the line of action of the force. The absolute value of $\overline{M}_0\left(\overline{F}\right)$ is (see Fig. 6.2).

$$\left|\overline{M}_0\left(\overline{F}\right)\right| = \left|\overline{r}\right| \cdot \left|\overline{F}\right| \cdot \sin\alpha = \left|\overline{F}\right| \cdot d\,, \tag{6.2}$$

that is to say, the product of the absolute value of the force $\overline{F}$ and the "arm" of the force with respect to point O. The "arm" is the length of the perpendicular from O onto the line of action of the force.

### 6.2.1. Moment of a sliding force

If the force $\overline{F}$ is sliding on its line of action from $A$ to $B$ (Fig. 6.3) the moment $\overline{M}_0\left(\overline{F}\right)$ does not change, because $\overline{OB} = \overline{OA} + \overline{AB}$, or $\overline{r}' = \overline{r} + \overline{AB}$ and

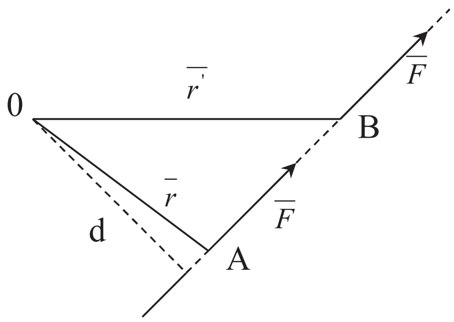


**Fig. 6.3 The moment of a sliding force**

$$\overline{r}' \times \overline{F} = \left(\overline{r} + \overline{AB}\right) \times \overline{F} = \overline{r} \times \overline{F} + \overline{AB} \times \overline{F} = \overline{r} \times \overline{F} = \overline{M}_0\left(\overline{F}\right) \tag{6.3}$$

($\overline{AB} \times \overline{F} = \overline{0}$, because $\overline{AB}$ and $\overline{F}$ are two parallel vectors).

### 6.2.2. Variation of the moment with the position of the pole

If the moment of a force is determined about another point (pole) denoted P, its value will change to:

$$\overline{M}_P\left(\overline{F}\right)=\overline{r}'\times\overline{F}=\left(\overline{PO}+\overline{r}\right)\times\overline{F}=\overline{PO}\times\overline{F}+\overline{M}_O\left(\overline{F}\right)=\overline{M}_O\left(\overline{F}\right)-\overline{OP}\times\overline{F} \quad (6.4)$$

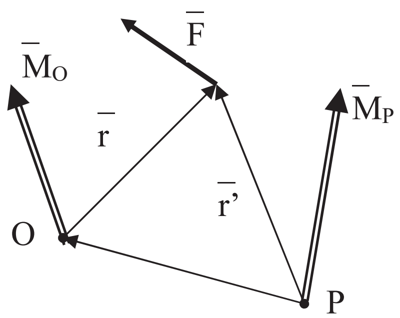


**Fig. 6.4. Variation of the moment with the point of evaluation**

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### 6.2.3. Characterization of a sliding force

If *X, Y, Z* are the projections of $\overline{r}$ and *X, Y, Z* are the projections of $\overline{F}$ on the axes of a Cartesian coordinate system *Oxyz*, the projections of $\overline{M}_0\left(\overline{F}\right)$ are:

$$\overline{M}_0\left(\overline{F}\right)=\overline{r}\times\overline{F}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ x & y & z\\ X & Y & Z\end{vmatrix}=\left(yZ-zY\right)\overline{i}+\left(zX-xZ\right)\overline{j}+\left(xY-yX\right)\overline{k}. \quad (6.5)$$

It follows that a sliding force $\overline{F}$ may be characterized by six scalars, the projections of $\overline{F}$ and the projections of $M\left(F\right)$, i.e.

$$X,\ Y,\ Z,\ M_{ox}=yZ-zY,\ \ M_{oy}=zX-xZ,\ \ M_{oz}=xY-yX. \quad (6.6)$$

It is easy to verify that these six scalars are not independent because they satisfy the identity:

$$XM_{ox}+YM_{oy}+ZM_{oz}=0. \quad (6.7)$$

This identity represents the property of orthogonality of vectors $\overline{M}_0\left(\overline{F}\right)$ and $\overline{F}$.

## *6.3. The moment of a force about an axis*

By definition the projection on a given axis $\left(\Delta\right)$ of the moment $\overline{M}_0\left(\overline{F}\right)$ of a force $\overline{F}$ with respect to point *O* on this axis is called moment of the force about the axis.

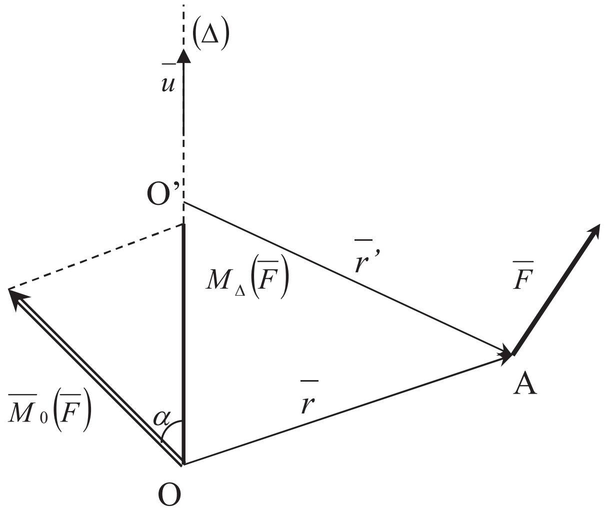



**Fig. 6.5 Moment of a force about an axis**

The value of the moment $M_\Delta\left(\overline{F}\right)$ is (see Fig. 6.5):

$$M_\Delta\left(\overline{F}\right)=\left|\overline{M}_0\left(\overline{F}\right)\right|\cdot\cos\alpha=\overline{M}_0\left(\overline{F}\right)\cdot\overline{u}=\left(\overline{r}\times\overline{F}\right)\cdot\overline{u}=\left(\overline{r},\overline{F},\overline{u}\right) \tag{6.8}$$

where $\overline{u}$ is the unit vector of the axis. If $\overline{r}(x,y,z)$, $\overline{F}(X,Y,Z)$ and $\overline{u}(\cos\alpha,\cos\beta,\cos\gamma)$ then $\overline{M}_0\left(\overline{F}\right)$ may be written in the form:

$$M_\Delta\left(\overline{F}\right)=\begin{vmatrix} x & y & z \\ X & Y & Z \\ \cos\alpha & \cos\beta & \cos\gamma \end{vmatrix}. \tag{6.9}$$

From the known proprieties of determinants it easily follows that:

$$\left(\overline{r},\overline{F},\overline{u}\right)=\left(\overline{u},\overline{r},\overline{F}\right)=\left(\overline{F},\overline{u},\overline{r}\right). \tag{6.10}$$

The moment of the force $\overline{F}$ about the axis $(\Delta)$ does not change if the point *O* is replaced by *O'*, because $\overline{r}'=\overline{O'A}=\overline{O'O}+\overline{OA}=\overline{O'O}+\overline{r}$ (see Fig. 6.5) or $\left(\overline{O'O}+\overline{r},\overline{F},\overline{u}\right)=\left(\overline{O'O},\overline{F},\overline{u}\right)+\left(\overline{r},\overline{F},\overline{u}\right)=\left(\overline{r},\overline{F},\overline{u}\right)=M_\Delta\left(\overline{F}\right)$;

The moment of the force about the axis $(\Delta)$, $M_\Delta\left(\overline{F}\right)=\left(\overline{r},\overline{F},\overline{u}\right)$ is null if the three vectors $\overline{r},\overline{F},\overline{u}$ are in the same plane i.e. if the line of action of the force and the axis are in the same plane. Three possibilities exist:

a) The line of action of the force intersects the axis;
b) The line of action of the force and the axis are parallel;
c) The line of the force and the axis coincide.

### 6.4. Elementary operations with sliding vectors

Elementary operations with sliding vectors are those operations which transform a system of sliding forces into another equivalent system of sliding forces. There are two such operations:

a) The sliding of a force along its support line (direction of action)
b) The replacement of a concurrent system of forces by another one having the same resultant force vector, defined in the next paragraph.

### 6.5. Resultant force vector. Resultant moment vector

By definition the **resultant force vector** is the vector sum of a given system of sliding forces. The **resultant moment vector** is the sum of the moments of all the forces of a given system of sliding forces with respect to a chosen point O, called **pole**. Their expressions are

$$\overline{R} = \sum_i \overline{F}_i ; \quad \overline{M}_o = \sum_i \overline{r}_i \times \overline{F}_i \,. \tag{6.11}$$

Note that in general it is incorrect to call these vectors "resultant force" or "resultant moment" since in general none of them is the global "resultant" but only together represent the resultant action on a given body.

It is easy to verify that the resultant force vector and the resultant moment vector are invariant to elementary operations on sliding forces.

### 6.6. Variation of the resultant moment vector as the pole changes

Consider another pole P to compute the moments of forces about it. Since $\overline{PA_i} = \overline{OA_i} + \overline{PO}$ or $\overline{r}_i' = \overline{r}_i + \overline{PO}$, it follows (Fig. 6.6) for the moment:

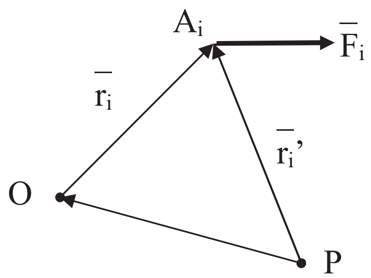



**Fig. 6.6 Change of pole for the moment of a force**

$$\begin{aligned}\overline{M}_P &= \sum_i \overline{r}_i' \times \overline{F}_i = \sum_i \left(\overline{r}_i + \overline{PO}\right) \times \overline{F}_i = \sum_i \overline{r}_i \times \overline{F}_i + \sum_i \overline{PO} \times \overline{F}_i \\ &= \overline{M}_o + \overline{PO} \times \sum_i \overline{F}_i\end{aligned} \tag{6.12}$$

Hence the moment about another pole changes by subtracting the moment of the resultant force vector from the moment already determined at the location of the initial pole:

$$\bar{M}_P = \bar{M}_O - \overline{OP} \times \bar{R}\,. \qquad (6.13)$$

The following corollaries are consequences of this formula:

a) If $\bar{R} = 0$ and $\bar{M}_O = 0$, then in any point P of the space $\bar{R} = \bar{0}$ and $\bar{M}_P = \bar{0}$

b) If $\bar{R} = 0$ and $\bar{M}_O \neq 0$, then in any point P of the space $\bar{M}_P = \bar{M}_O$. In this case the resulting moment vector is a free vector.

c) If the new pole P belongs to a segment OP parallel to $\bar{R}$, then $\overline{OP} \times \bar{R} = \bar{0}$ and the resulting moment vector remains unchanged: $\bar{M}_P = \bar{M}_O$.

d) The scalar product between the resulting force vector and the resulting moment vector is a constant.

Indeed, if the projections of the two vectors are $\bar{R} = X\bar{i} + Y\bar{j} + Z\bar{k}$ and $\bar{M}_O = M_{Ox}\bar{i} + M_{Oy}\bar{j} + M_{Oz}\bar{k}$, by scalar multiplication of (6.12) by $\bar{R}$, one gets

$$\bar{R} \cdot \bar{M}_P = \bar{R} \cdot \bar{M}_O - \bar{R} \cdot \left(\overline{OP} \times \bar{R}\right) = \bar{R} \cdot \bar{M}_O = XM_{Ox} + YM_{Oy} + ZM_{Oz} \qquad (6.14)$$

The expression $\bar{R} \cdot \bar{M}_O = XM_{Ox} + YM_{Oy} + ZM_{Oz}$ is called the **invariant trinomial**.

e) The projection of the resultant moment vector along the direction of the resultant force vector is a constant.

Indeed the projection made at a pole P and at pole O are

$$\frac{\bar{R}}{|\bar{R}|} \cdot \bar{M}_P = |\bar{M}_P| \cos\beta; \quad \frac{\bar{R}}{|\bar{R}|} \cdot \bar{M}_O = |\bar{M}_O| \cos\alpha \qquad (6.15)$$

and from (6.14) it follows

$$|\bar{M}_P| \cos\beta = |\bar{M}_O| \cos\alpha = M_r\,. \qquad (6.16)$$

The common projection is

$$M_r = \bar{M}_O \frac{\bar{R}}{|\bar{R}|} = \frac{XM_{Ox} + YM_{Oy} + ZM_{Oz}}{\sqrt{X^2 + Y^2 + Z^2}}\,. \qquad (6.17)$$

### *6.7. Central axis of a system of sliding forces*

The question of finding the locus (geometric location) of poles about which the resulting moment vector is a minimum will be addressed in the following. The resulting force vector $\bar{R} = X\bar{i} + Y\bar{j} + Z\bar{k}$ and the resulting moment vector $\bar{M}_O = M_{Ox}\bar{i} + M_{Oy}\bar{j} + M_{Oz}\bar{k}$ are considered already determined with respect to a coordinate system having O as origin. The resulting moment vector about another pole P(x,y,z) is

$$\overline{M}_P = \overline{M}_O - \overline{OP} \times \overline{R} = M_{Ox}\overline{i} + M_{Oy}\overline{j} + M_{Oz}\overline{k} - \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ x & y & z \\ X & Y & Z \end{vmatrix}$$

$$= \left(M_{Ox} - yZ + zY\right)\overline{i} + \left(M_{Oy} - zX + xZ\right)\overline{j} + \left(M_{Oz} - xY + yX\right)\overline{k} \tag{6.18}$$

It has been proven that the projection of $\overline{M}_P$ along the direction of $\overline{R}$ is a constant.

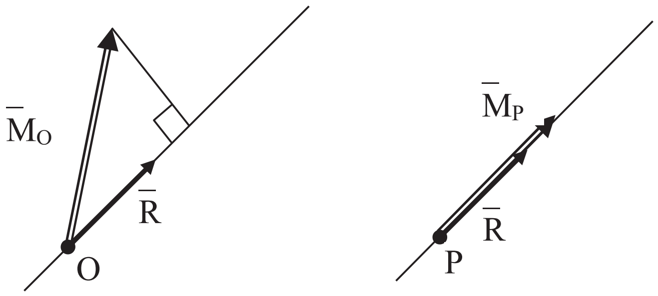



**Fig. 6.7 Projection of the resulting moment vector along the resulting force vector**

As shown in Fig. 6.7, if the projection along a given direction is to remain constant, the minimum resulting moment vector having that projection is one having its modulus exactly equal with the projection. This can only be possible if the resulting moment vector is parallel to the direction of $\overline{R}$. This last conclusion can be expressed as $\overline{M}_P \| \overline{R}$ or, by the proportionality of the scalar projections from (6.16) and those of $\overline{R}$:

$$\frac{M_{Ox} - yZ + zY}{X} = \frac{M_{Oy} - zX + xZ}{Y} = \frac{M_{Oz} - xY + yX}{Z} \tag{6.19}$$

This last formula represents the canonical equation of a straight line. The points in space having the requested minimizing property have been thus proven to belong to a line, called **central axis**. Another conclusion from (6.17) and (6.19) is that the minimum obtainable moment is $M_r$.

## *6.8. Reduction of systems of sliding forces*

By computing the resulting force vector and the resulting moment vector, one of the next four cases can exist:

1) $\overline{R} = \overline{0}$ and $\overline{M}_O = \overline{0}$: The system is equivalent to zero (no) acting forces.

2) $\overline{R} \neq \overline{0}$ and $\overline{M}_O = \overline{0}$: The system is equivalent to a resulting force passing through the origin O of the chosen system of coordinates.

3) $\overline{R} = \overline{0}$ and $\overline{M}_O \neq \overline{0}$: The system is equivalent to couple of forces acting in a plane perpendicular on the obtained resulting moment vector.

4) $\overline{R} \neq \overline{0}$ and $\overline{M}_O \neq \overline{0}$: Two possibilities exist in this case, depending on the projection of $\overline{M}_O$ along $\overline{R}$. This projection is directly related to the scalar product of the two vectors, so the possibilities are:

a) $\overline{R} \cdot \overline{M}_O = 0$. Means that there are points in space about which the minimum moment is zero. In other words the system is equivalent with a single force $\overline{R}$, only placed on the central axis. In this case $\overline{R}$ is called **resulting force** because it replaces the whole system of sliding forces. If $\overline{r}$ is a position vector reaching the central axis, then the resulting moment vector is $\overline{M}_O = \overline{r} \times \overline{R}$. In this case applies the **Varignon theorem**: "The resulting moment vector equals the moment of the resultant force".

b) $\overline{R} \cdot \overline{M}_O \neq 0$. The system is equivalent to a **wrench**, that is a force $\overline{R}$ acting along the central axis of the system and a couple of forces acting in a plane perpendicular to that axis, having as modulus

$$M_r = \overline{M}_O \frac{\overline{R}}{|\overline{R}|} = \frac{XM_{Ox} + YM_{Oy} + ZM_{Oz}}{\sqrt{X^2 + Y^2 + Z^2}}. \tag{6.20}$$

## *6.9. Particular systems of sliding vectors*

### 6.9.1. A system of concurrent forces

The systems of forces have their directions concurrent in a point O. It follows that

$$\overline{R} = \sum_i \overline{F}_i; \quad \overline{M}_O = \overline{0} \tag{6.21}$$

A system of concurrent forces is equivalent either with a resulting force $\overline{R} \neq \overline{0}$ or is equivalent with no acting force $\overline{R} = \overline{0}$.

### 6.9.2. A system of coplanar forces

The forces are all included in a plane. Be it the Oxy plane. It follows that:

$$\begin{aligned} \overline{R} &= \sum_i \overline{F}_i = \left(\sum_i X_i\right)\overline{i} + \left(\sum_i Y_i\right)\overline{j}; \\ \overline{M}_O &= \sum_i \overline{r}_i \times \overline{F}_i = \sum_i \left(x_i Y_i - y_i X_i\right)\overline{k} \end{aligned} \tag{6.22}$$

It follows that in any case $\overline{R} \cdot \overline{M}_O = 0$. So, the system of coplanar forces can be equivalent with:

- No acting forces if $\overline{R} = \overline{0}$ and $\overline{M}_O = \overline{0}$
- A resultant force if $\overline{R} \neq \overline{0}$.
- A couple of forces if $\overline{R} = \overline{0}$ and $\overline{M}_O \neq \overline{0}$.

### 6.9.3. A system of couples of forces

A system of couples of forces can be seen as pairs of parallel, opposite forces having the same magnitude. Consequently,

$$\overline{R} = \overline{0};\ \overline{M}_O = \sum_i \overline{M}_i \tag{6.23}$$

Such a system of couples can be made equivalent with:
- No acting forces if $\overline{R} = \overline{0}$ and $\overline{M}_O = \overline{0}$
- A couple of forces $\overline{M}_O = \sum_i \overline{M}_i$ .

### 6.9.4. A system of parallel forces

All the forces in the system are parallel with a direction given by unit vector $\overline{u}$ which in general does not coincide with any of the axes of a chosen coordinate system.
It follows that

$$\begin{aligned} \overline{R} &= \sum_i \overline{F}_i = \left(\sum_i F_i\right)\overline{u}; \\ \overline{M}_O &= \sum_i \overline{r}_i \times \overline{F}_i = \sum_i \overline{r}_i \times F_i\overline{u} = \left(\sum_i F_i\overline{r}_i\right) \times \overline{u} \end{aligned} \tag{6.24}$$

In any case $\overline{R} \cdot \overline{M}_O = 0$ because $\overline{R}$ has been proven to be parallel to $\overline{u}$ and $\overline{M}_O$ is perpendicular to $\overline{u}$ .
Such a system of forces can be equivalent with one of the following:
- No acting forces if $\overline{R} = \overline{0}$ and $\overline{M}_O = \overline{0}$ .
- A resultant force if $\overline{R} \neq \overline{0}$ either passing through the origin or placed along the central axis.
- A couple of forces if $\overline{R} = \overline{0}$ and $\overline{M}_O \neq \overline{0}$ .

### 6.9.5. Center of parallel forces

Consider the case $\overline{R} \neq \overline{0}$ for a system of parallel forces. The central axis can be determined using the theorem of Varignon, since in this case $\overline{R} \cdot \overline{M}_O = 0$ .

$$\overline{M}_O = \left(\sum_i F_i\overline{r}_i\right) \times \overline{u} = \overline{r} \times \overline{R} = \overline{r} \times \left(\sum_i F_i\right)\overline{u} = \left(\sum_i F_i\right)\overline{r} \times \overline{u} \tag{6.25}$$

Taking the first and last expression of these equalities and passing both expressions on one side, it can be written:

$$\left[\left(\sum_i F_i\overline{r}_i\right) - \left(\sum_i F_i\right)\overline{r}\right] \times \overline{u} = \overline{0} . \tag{6.26}$$

This formula states that in the general case the two vectors involved in the vector product are parallel, fact that can also be expressed as:

$$\left[\left(\sum_i F_i\overline{r}_i\right)-\left(\sum_i F_i\right)\overline{r}\right]=\lambda\overline{u}\,, \tag{6.27}$$

in which λ is a scalar parameter.

Another scalar parameter can be defined as $\mu=-\dfrac{\lambda}{\sum_i F_i}$, which allows to write the vector position of an arbitrary point of the central axis:

$$\overline{r}=\frac{\left(\sum_i F_i\overline{r}_i\right)}{\sum_i F_i}+\mu\overline{u}\,. \tag{6.28}$$

The first term represents the position of a point of the axis. Its position vector is

$$\overline{\rho}=\frac{\left(\sum_i F_i\overline{r}_i\right)}{\sum_i F_i}\,. \tag{6.29}$$

As it can be easily seen, the central axis will pass through this point for any $\overline{u}$. This means that if the forces applied in each point will all change direction remaining parallel, the new central axis will pass through the same point. For this reason this point is called **the center of parallel forces**.

The coordinates of the center of parallel forces are, as a consequence of (6.29):

$$\xi=\frac{\sum_i F_i x_i}{\sum_i F_i};\quad \eta=\frac{\sum_i F_i y_i}{\sum_i F_i};\quad \zeta=\frac{\sum_i F_i z_i}{\sum_i F_i}\,. \tag{6.30}$$

A useful application is to determine the center of gravity. If the gravity forces applied to a system of material points can be assumed to be parallel, then $\overline{F}_i=\overline{G}_i=m_i\overline{g}$ and the above formulas can be applied. The **center of gravity** is then

$$\xi=\frac{\sum_i m_i x_i}{\sum_i m_i};\quad \eta=\frac{\sum_i m_i y_i}{\sum_i m_i};\quad \zeta=\frac{\sum_i m_i z_i}{\sum_i m_i}\,. \tag{6.31}$$

Compared with the mass center, the coordinates are identical in the assumed hypothesis.

### *6.10. Distributed forces*

In most technical applications, forces are applied not in a point but on an area. Examples are snow on a roof, forces produced by air/water pressure applied on a body, etc. For two-dimensional applications, this corresponds to application of a force along a line of arbitrary shape (Fig. 6.8).

The distributed force in a two-dimensional application is represented by p(x) measured in N/m and in general this distributed force is not normal to the line, making an angle θ with the reference direction (Ox in Fig. 6.8b). Pressure and contact forces are however normal to the surfaces and friction or viscous drag force are tangent to them. The problem is to find the equivalent system for the distributed force.

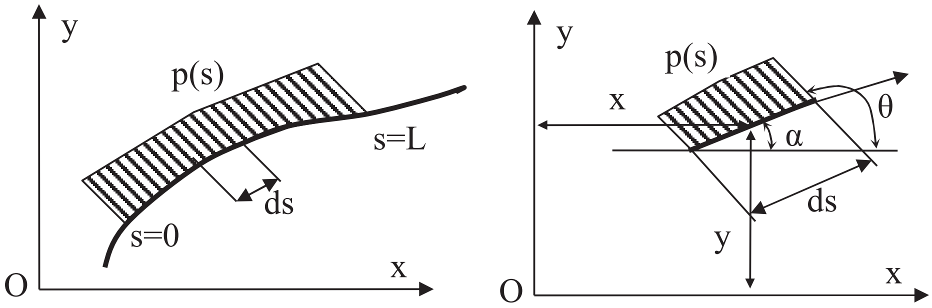
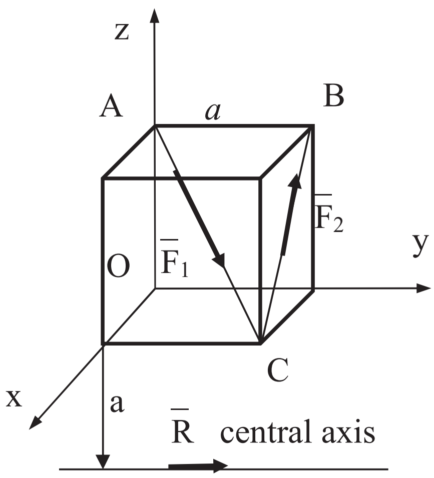



**Fig. 6.8 Distributed force**

The elementary forces and moment applied on an elementary arc of length ds (Fig. 6.8b) are

$$dX = -p(s)\cos\big(\theta(s)\big)ds$$
$$dY = -p(s)\sin\big(\theta(s)\big)ds \qquad (6.32)$$
$$dM_{Oz} = xdY - ydX;$$

The resulting force vector has two components and the moment one projection:




$$X = -\int_0^L p(s)\cos\big(\theta(s)\big)ds$$

$$Y = -\int_0^L p(s)\sin\big(\theta(s)\big)ds$$

$$M_{Oz} = \int_0^L p(s)\big[y(s)\cos\theta(s) - x(s)\sin\theta(s)\big]ds \qquad (6.33)$$

<u>Example 1</u>.

Consider two forces of modulus

$|F_1| = P\sqrt{3}$ and $|F_2| = P\sqrt{2}$ acting as shown in the figure, on a cube of edge $a$. Find the equivalent system.

The projections of the forces are:

$$\overline{F}_1 = |\overline{F}_1| \frac{\overline{AC}}{|\overline{AC}|} = P\sqrt{3}\frac{a\overline{i} + a\overline{j} - a\overline{k}}{\sqrt{3a^2}} = P\overline{i} + P\overline{j} - P\overline{k}$$

$$\overline{F}_2 = |\overline{F}_2| \frac{\overline{CB}}{|\overline{CB}|} = P\sqrt{2}\frac{-a\overline{i} + 0\overline{j} + a\overline{k}}{\sqrt{2a^2}} = -P\overline{i} + P\overline{k}$$

The moments produced by the forces are:

$$\overline{M}_O(\overline{F}_1) = \overline{OA} \times \overline{F}_1 = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & a \\ P & P & -P \end{vmatrix} = -aP\overline{i} + aP\overline{j}$$

$$\overline{M}_O(\overline{F}_2) = \overline{OC} \times \overline{F}_1 = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a & a & 0 \\ -P & 0 & P \end{vmatrix} = aP\overline{i} - aP\overline{j} + aP\overline{k}\ .$$

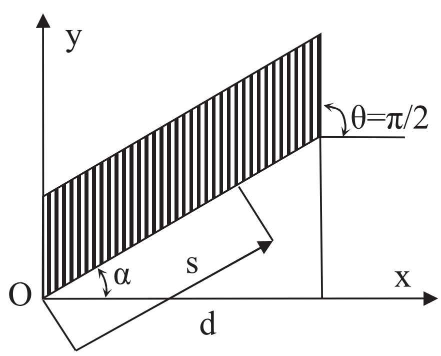
The effect in O is a "wrench" : $\begin{cases} \overline{R} = \overline{F}_1 + \overline{F}_2 = P\overline{j}; \\ \overline{M}_O = aP\overline{k} \end{cases}$

The scalar product is $\overline{R} \cdot \overline{M}_O = 0$ so the system is equivalent with a force $\overline{R}$ on the central axis:

$\dfrac{aP - y0 + zP}{0} = \dfrac{-aP}{P} = \dfrac{aP - xP}{0}$ which is the intersection of planes $z = -a;\ x = a$ shown also on the figure.

Example 2.

A uniformly distributed vertical force is applied downwards on a plane inclined by an angle $\alpha$, covering a distance $d$ on the horizontal. Find the resulting force and moment vectors in O.




$$X = -\int_0^L p\cos(\pi/2)\,ds = 0$$

$$Y = -\int_0^L p\sin(\pi/2)\,ds = -pL = -p\frac{d}{\cos\alpha}$$

$$M_{Oz} = \int_0^L p\left[-s\cos\alpha\sin(\pi/2)\right]ds - p\cos\alpha\left.\frac{s^2}{2}\right|_0^L$$

$$= -p\frac{d^2}{2\cos\alpha} = Y\frac{d}{2}$$

It has been used the obvious fact: $x = s\cos\alpha$.

# 7. STATICS OF A RIGID BODY

## 7.1. Free rigid body

A rigid body is said to be free if the possible motions of this body are not subject to any constraints. A free rigid body has six degrees of freedom: three translations along the axes *Ox, Oy* and *Oz*, and three rotations about these axes (Fig. 7.1).

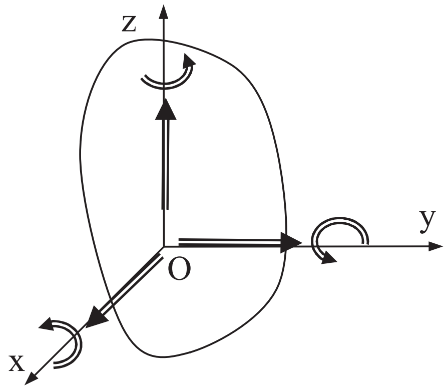



**Fig. 7.1 Possible motions of a rigid body**

A system of sliding forces acting on a rigid body is said to be in equilibrium if the rigid body, initial at rest, continues to be at the rest, afterwards. A system of sliding forces is in equilibrium if and only if both, the resultant force vector and the resultant moment vector are zero, i.e. if the system of sliding forces is equivalent to a zero force. As proof, obviously a zero force has no effect on a rigid body. Therefore the condition is sufficient. It is easy to verify experimentally that a rigid body under the action of a unique force, or of a couple, or of a wrench, cannot remain at rest. Therefore the condition is also necessary.
This condition may be written:

$$\sum_{i=1}^{n} \overline{F}_i = \overline{0};\ \sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i = \overline{0} \tag{7.1}$$

The projections of these equations on the axes of a Cartesian frame are:

$$\begin{gathered}\sum_{i=1}^{n} X_i = 0;\ \sum_{i=1}^{n} Y_i = 0;\ \sum_{i=1}^{n} Z_i = 0\\ \sum_{i=1}^{n} M_{ix} = 0;\ \sum_{i=1}^{n} M_{iy} = 0; \sum_{i=1}^{n} M_{iz} = 0\end{gathered} \tag{7.2}$$

If the forces lie in Oxy - plane, (7.2.) becomes:

$$\sum_{i=1}^{n} X_i = 0;\ \sum_{i=1}^{n} Y_i = 0;\ \sum_{i=1}^{n} M_{iz} = 0\ . \tag{7.3}$$

If the forces are parallel to *Oz* - axis, (7.2) becomes:

$$\sum_{i=1}^{n} Z_i = 0; \sum_{i=1}^{n} M_{ix} = 0;\ \sum_{i=1}^{n} M_{iy} = 0\,. \tag{7.4}$$

For a system of forces couples, (7.2) becomes

$$\sum_{i=1}^{n} M_{ix} = 0; \sum_{i=1}^{n} M_{iy} = 0; \sum_{i=1}^{n} M_{iz} = 0 \, . \tag{7.5}$$

Consider a free rigid body and $N$ forces of magnitudes $F_1,...,F_N$ respectively acting on it. Using the theory of sets of straight lines (Annex), the equations (7.2) may be written

$$\begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_N \\ b_1 & b_2 & \cdots & \cdots & b_N \\ c_1 & c_2 & \cdots & \cdots & c_N \\ l_1 & l_2 & \cdots & \cdots & l_N \\ m_1 & m_2 & \cdots & \cdots & m_N \\ n_1 & n_2 & \cdots & \cdots & n_N \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} . \tag{7.6}$$

where $a_i$, $b_i$, $c_i$, $l_i$, $m_i$, $n_i$ $(i=1,2,...,N)$ are the homogeneous coordinates of the lines of action of the forces. This homogeneous linear system of equations may have non zero solutions for $F_1,...,F_N$ if and only if the rank of rectangular matrix r<N.

It follows then as necessary (but not sufficient) conditions (See also the Annex):

a) If N=1 the equilibrium is impossible.

b) Two forces must have the same line of action.

c) Three forces must have their lines of action situated in a plane and these lines must be concurrent or parallel.

d) Four forces must have their lines of action into the same family of rectilinear generatrices of a rectilinear quadric; in particular these lines of action may be concurrent of parallel.

e) Five forces must have their lines of action into the same linear congruence; in particular these lines of action may intersect two arbitrary straight lines.

f) Six forces must have their lines of action into the same linear complex; in particular these lines of action may intersect a given straight line or may be parallel to a given plane.

### *7.2. The constrained rigid body*

A rigid body is said to be constrained if its motions are subject to certain conditions. These conditions are called **constraints**. By virtue of the **principle of constraints** it shall be assumed that besides the given forces, the constrained rigid body is acted upon by additional forces called **reactions** which cause the body to comply with the constraints. The reactions are generated from those bodies which limit the freedom of motion for the given constrained rigid body. The other forces acting on a constrained rigid body will be called **active forces** (in order to differentiate them from reactions). If the reactions are added to the active forces, then the constrained rigid body can be considered as free.

### *7.3. Smooth constraints of a rigid body*

#### 7.3.1. The simple support

The simple support (or movable support) is a constraint generated by a certain body supporting another (Fig. 7.2.a ).

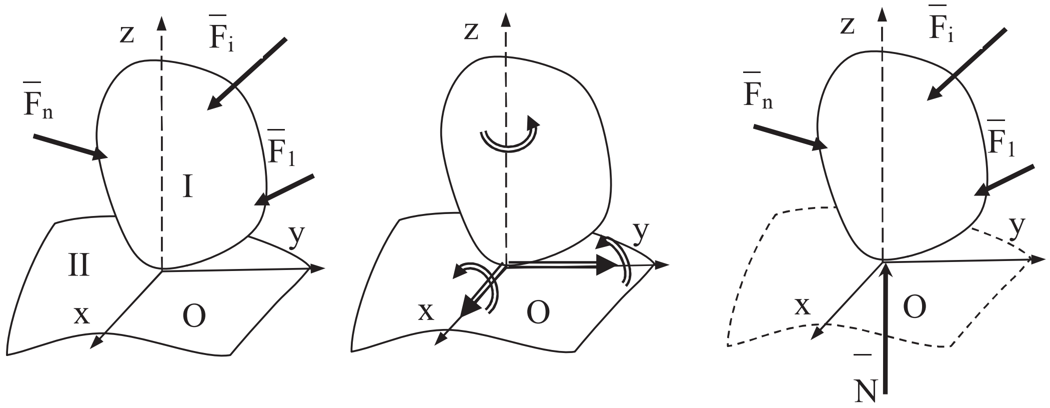



**Fig. 7.2 Simple support. a) Two bodies b) Possible motions c) Normal reaction**

Let O be the point of contact of a given rigid body (I) with another rigid body (II). The plane Oxy is the common tangent plane and *Oz* is the common normal in *O* to the surfaces of the two bodies. Fig. 7.2b shows the possible mobility of the rigid body (I) with respect to the rigid body (II): two independent translations along *Ox* and *Oy* axes and three independent rotations about *Ox, Oy* and *Oz* axes. Fig. 7.2c shows the reaction which is called **normal reaction**. If the constraint is smooth (without friction) this is the only reaction corresponding to the obstructed mobility, which is the translations along Oz -axis.

#### 7.3.2. The articulation joint

The articulation joint is a type of constraint joining two bodies, which allows their relative rotations about a point (spherical joint) or about an axis (hinge).
If the articulation joint is a spherical joint (Fig. 7.3a), then the given rigid body (I) has three independent rotation about *Ox, Oy* and *Oz* axes with respect to the rigid body (II).
b) Fig. 7.3c shows the reactions $R_x$, $R_y$ and $R_z$ corresponding to the three obstructed motions, the translations along the *Ox, Oy* and respectively *Oz* axes.
If the articulation joint is a hinge and the lines of action of the forces acting on the given rigid body (I) lie in the Oxy-plane (Fig. 7.4a) then the given rigid body (I) has a single possible rotation about the *Oz* axis, normal to the figure (Fig. 7.4b).

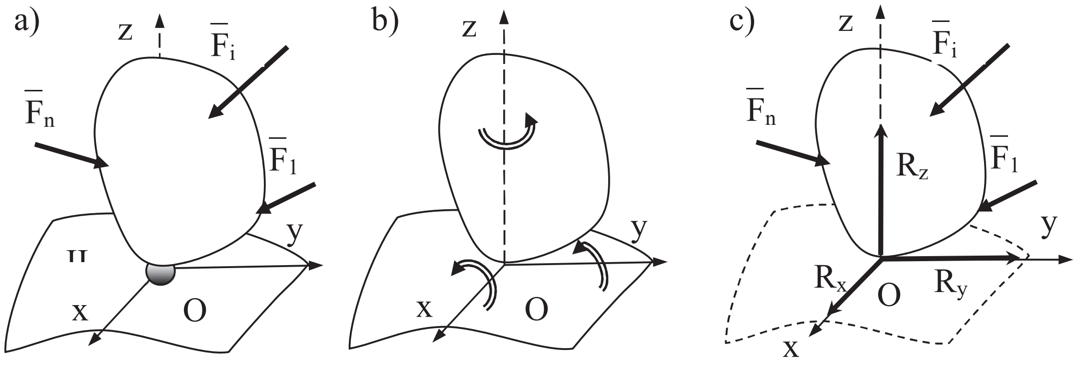



**Fig. 7.3 Spherical articulation joint (a). Possible motions (b). Reactions (c)**

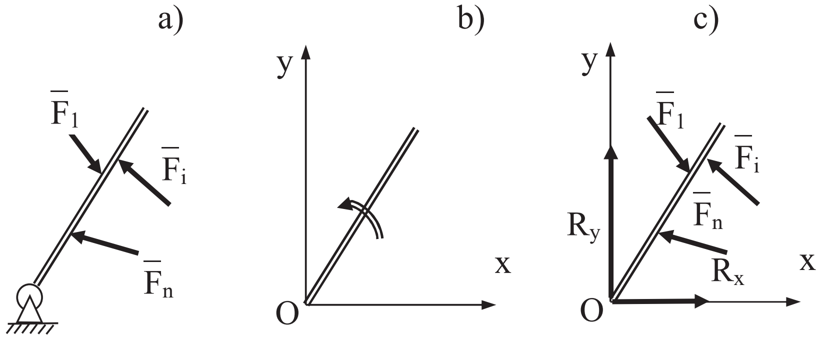



**Fig. 7.4. Hinge. Forces and symbol (a). Possible motion (b). Reactions (c)**

Fig. 7.4c shows the reactions $R_x$ and $R_y$ corresponding to the obstructed motions, the translations along the *Ox* and respectively *Oy* axes.

### 7.3.3. The rigid fixing

The rigid fixing (or built in mounting) of a beam for example is the fixing of a beam in such a way that the section at the plane of fixing, is not subjected to any rotation or translation (Fig. 7.5a).

The given rigid body (I) has no possible motions (Fig. 7.5b). Fig. 7.5c shows the reactions which are three force projections $R_x$, $R_y$ and $R_z$ corresponding to the three obstructed displacements along the Ox, Oy and respectively Oz axes and three projections of a reaction moment $M_{Ox}$, $M_{Oy}$ and $M_{Oz}$ corresponding to the three obstructed rotations about the *Ox, Oy* and respectively *Oz* axes.

If the forces acting on the given rigid body (I) lie in the Oxy plane (Fig. 7.6a), then the reaction are (Fig. 7.6b) the forces $R_x$ and $R_y$ corresponding to the two obstructed translations along Ox and Oy axes and the couple $M_O$ corresponding to the obstructed rotation about the Oz axis.

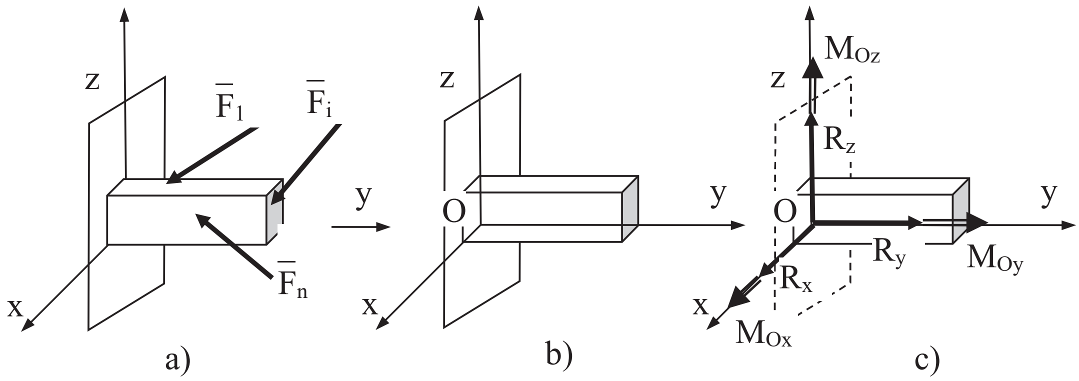



**Fig. 7.5 Rigid fixing (a). Possible motions: all blocked (b). Reactions (c)**

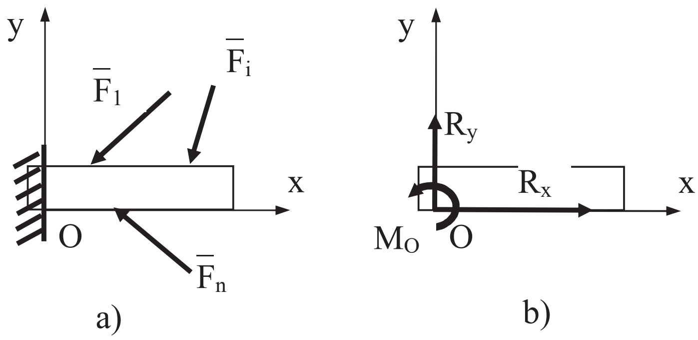



**Fig. 7.6 Two dimensional rigid fixing (a). Reactions (b)**

## *7.4. A general theory for smooth constraints of a rigid body*

It is possible to formulate a general theory of smooth constraints of a rigid body, using the concepts of work and power. The elementary work of a force $\bar{R}$ is, by definition, the scalar product:

$$\bar{R}d\bar{r} = R_x dx + R_y dy + R_z dz \quad (7.7)$$

and the power is the time rate of work:

$$\bar{R}\bar{v} = R_x v_x + R_y v_y + R_z v_z \,. \quad (7.8)$$

By analogy, the expression for the power produced by the moment of a force is

$$\bar{M}_O \bar{\omega} = M_{Ox}\omega_x + M_{Oy}\omega_y + M_{Oz}\omega_z \,. \quad (7.9)$$

A constraint of a rigid body is smooth if and only if the reactions $R_x$, $R_y$, $R_z$, $M_{Ox}$, $M_{Oy}$ and $M_{Oz}$ and the velocities $v_x$, $v_y$, $v_z$, $\omega_x$, $\omega_y$, $\omega_z$ satisfy as an identity the condition:

$$R_x v_x + R_y v_y + R_z v_z + M_{Ox}\omega_x + M_{Oy}\omega_y + M_{Oz}\omega_z = 0 \,. \quad (7.10)$$

Some examples of smooth constraints of a rigid body are presented on Fig. 7.7.

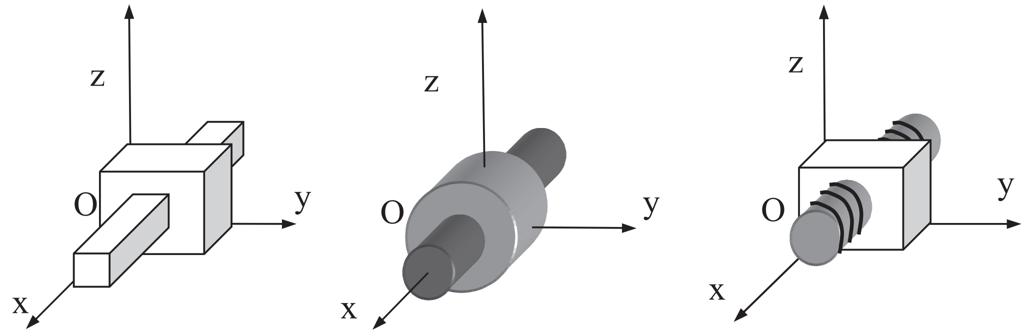



**Fig. 7.7 Examples of smooth constraints**

The slider joint (Fig. 7.7a) allows only a translation about the *Ox* axis. The only zero reaction is $R_x=0$, but the velocity $v_x$ is arbitrary. There are no other displacements, so that formula (7.10) is verified.
For the cylindrical joint (Fig. 7.7b) which allows a relative rotation about the Ox – axis and an independent relative translation in the direction of this axis:
$R_x = 0$, $M_{Ox}=0$, $v_y=0$, $v_z=0$, $\omega_y=0$, $\omega_z=0$ and again the formula is verified.
For the helical joint (Fig. 7.7c) which allows a relative screw motion $v_x=k.\omega_x$, $v_y=0$, $v_z=0$, $\omega_y=0$, $\omega_z=0$, so that (7.10) becomes: $\left(R_x k + M_{Ox}\right)\omega_x = 0$ relating the axial force in the screw to the applied moment.

Example 1.
Determine the reactions for a beam freely supported at its ends, having a span $l=10a$, carrying the loads shown in Fig. 7.8.

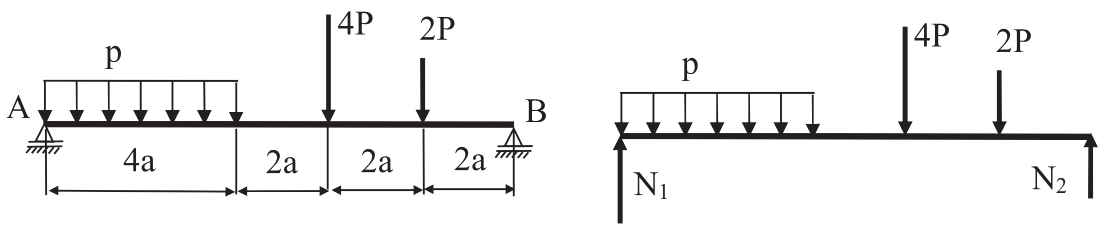



**Fig. 7.8 A bar acted by distributed and concentrated forces.**

Two normal reactions $N_1$ and $N_2$ are replacing the two simple supports. The conditions of equilibrium are:

$$\left(\sum X_i = 0\right): 0 = 0$$
$$\left(\sum Y_i = 0\right): N_1 + N_z - p4a - 4P - 2P = 0 \qquad (7.11)$$
$$\left(\sum M_{iA} = 0\right): N_2 \cdot 10a - p \cdot 4a \cdot 2a - 4P \cdot 6a - 2P \cdot 8a = 0$$

The values of the unknown reactions $N_1$ and $N_2$ are $N_1 = 2P+32pa$, $N_2 = 4p+0.8pa$.

Example 2.
Determine the reactions for a rod which is simply supported in B and hinged at A. It has a weight G applied at its middle (Fig. 7.9).

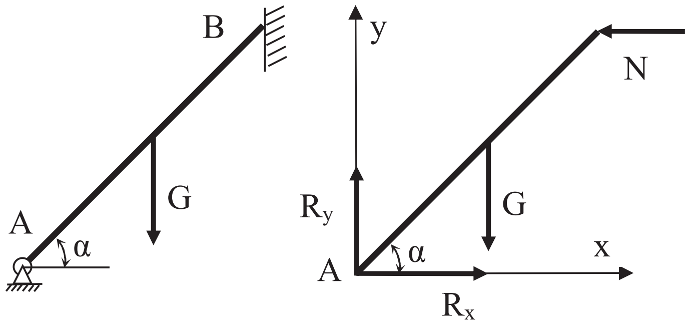



**Fig. 7.9 A hinged and supported rod**

The reactions $R_x$ and $R_y$ are replacing the hinge at A and the reaction N replaces the support in B. The conditions of equilibrium are:

$$\left(\sum X_i = 0\right): R_x - N = 0$$
$$\left(\sum Y_i = 0\right): R_y - G = 0 \qquad (7.12)$$
$$\left(\sum M_{iA} = 0\right): N \cdot l \sin\alpha - G \cdot \frac{l}{2} \cdot \cos\alpha = 0$$

The values of the unknowns $R_x$ ,$R_y$ and $N$ are: $R_x = N = \dfrac{G}{2\tan\alpha}$; $R_y = G$.

## *7.5. Rough constraints of a rigid body*

### 7.5.1. The sliding friction

Two bodies have rough surfaces and are in contact in a common tangency point. If a tendency to slide occurs between these surfaces, **sliding friction** occurs (Fig. 7.10). As in the case of a material point, the sliding friction may be replaced by a force $\overline{T}$ tangent to the two bodies, lying in the common tangent plane Oxy.

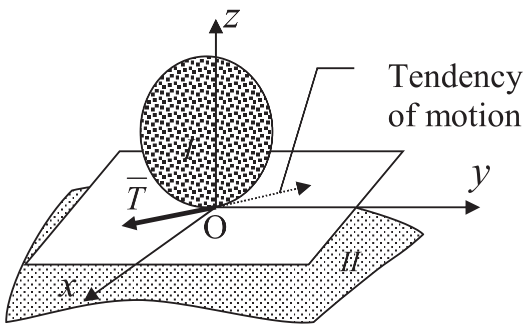



**Fig. 7.10. Sliding friction force between two rigid bodies**

The orientation of the **sliding friction force** $\bar{T}$ acting on the given body (I) is opposite to the sliding tendency of this body with respect to body (II). The magnitude $|\bar{T}|$ is limited to:

$$\left|\bar{T}\right| \le \mu\left|\bar{N}\right|, \tag{7.13}$$

in which $\mu$ is the sliding friction coefficient for the given pair of bodies (I) and (II) at the point of contact O.

Example.

A ladder of weight $G$ is resting on a rough floor (coefficient of sliding friction $\mu_1$) and against a vertical rough vertical wall (coefficient of sliding friction $\mu_2$). Determine the angle $\alpha$ for which the ladder is at equilibrium (Fig. 7.11).

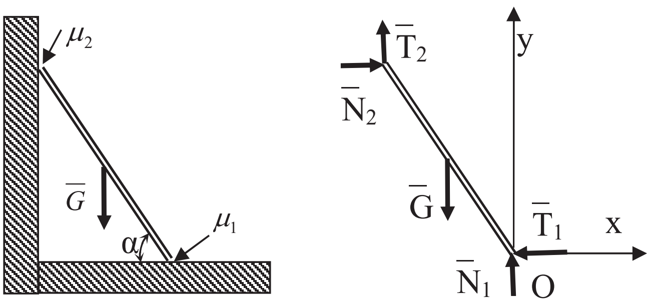



**Fig. 7.11 A ladder in rough contact with two surfaces**

The simple supports A and B can be replaced by the normal reactions $N_1$ and $N_2$ and by the friction forces $T_1$ and $T_2$. The equilibrium equations are:

$$\begin{cases} \sum_{i=1}^{3} X_i = 0: \; N_2 - T_1 = 0 \\ \sum_{i=1}^{3} Y_i = 0: \; N_1 + T_2 - G = 0 \\ \sum_{i=1}^{3} M_{iO} = 0: \; G\frac{l}{2}\cos\alpha - N_2 l \sin\alpha - T_2 l \cos\alpha = 0 \end{cases}$$

and the friction forces are limited by the inequalities: $T_1 \le \mu_1 N_1$; $T_2 \le \mu_2 N_2$. From the first three equations it follows that

$$T_1 = N_2; \quad T_2 = \frac{G}{2} - N_2 \tan\alpha; \quad N_1 = \frac{G}{2} + N_2 \tan\alpha;$$

The last two inequalities become:

$$N_2 \le \mu_1\left(\frac{G}{2} + N_2 \tan\alpha\right); \quad \frac{G}{2} - N_2 \tan\alpha \le \mu_2 N_2 \text{ or}$$

$$N_2\left(1 - \mu_1 \tan\alpha\right) \le \mu_1 \frac{G}{2}; \quad \frac{G}{2} \le N_2\left(\mu_2 + \tan\alpha\right)$$

Eliminating $N_2$ it follows for the angle α the condition: $tg\alpha \ge \frac{1-\mu_1\mu_2}{2\mu_1}$, which shows that the equilibrium is more sensitive to the friction coefficient with the horizontal surface ($\mu_1$) than to that of the vertical surface ($\mu_2$) .

### 7.5.2. The rolling friction

The rolling friction is the friction occurring when a real deformable body is rolling. The rolling motion can be considered as an instantaneous rotation about an axis lying in the common tangent plane, so the rolling friction may be replaced by a resistant couple, whose moment vector $\bar{M}_r$ acts along this axis and whose sense is inverse to the rolling tendency of the given body (I) with respect to the body (II). The magnitude is limited by

$$\left|\bar{M}_r\right| \leq s\left|\bar{N}\right|, \tag{7.14}$$

where $s$ is by definition, the static **coefficient of rolling friction** for a given pair of bodies (I) and (II) at the point of contact O.

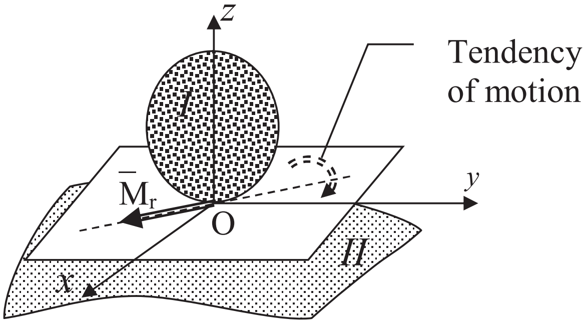



**Fig. 7.12. Rolling friction moment and tendency of motion**

Note that the static coefficient of the rolling friction $s$ is measured in units of length, while the coefficient of the sliding friction $\mu$ is a real number. The physical explanation of this moment is the asymmetrical deformation of the body and the surface, which in turn shifts the line of action of the normal reaction by a maximum distance $s$. Since the bodies must be rigid according to the hypothesis from the first chapter, this effect is taken into account by introducing the rolling friction moment.

Example 1. The pulled wheel.
A wheel of weight $\bar{G}$ and radius R is pulled by a horizontal force $\bar{F}$. The coefficients of sliding friction and rolling friction are respectively $\mu$ and s. Determine the magnitude of the applied force $\bar{F}$ for equilibrium condition (Fig. 7.13).

The simple support A is replaced by a normal reaction $\bar{N}$, a tangential force which is the sliding friction $\bar{T}$, opposite to the sliding tendency and the resistant couple of the rolling friction $\bar{M}_r$ opposite to the obvious rolling tendency.

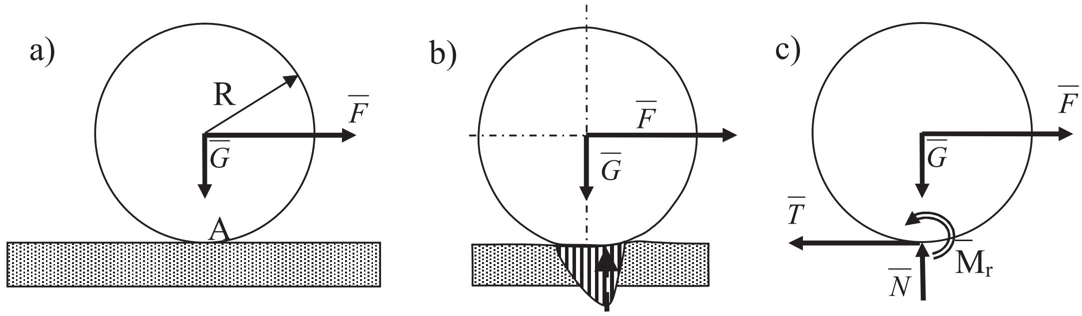



**Fig. 7.13 The equilibrium of a pulled wheel. Acting forces (a), distributed normal reaction (b), free body diagram (c)**

The conditions for equilibrium are:

$$\sum_{i=1}^{2} X_i = 0: \; F - T = 0; \qquad \sum_{i=1}^{2} Y_i = 0: \; N - G = 0; \qquad \sum_{i=1}^{2} M_{io} = 0: \; M_r - FR = 0$$

and the inequalities, valid in conditions of equilibrium: $T \le \mu N$; $M_r \le sN$.
From the first three equations, it follows that $F \le \mu G$ and $FR \le sG$.

If $s/R \le \mu$ then equilibrium is possible for $F \le \frac{s}{R} G$.
If $s/R \ge \mu$ then equilibrium is possible for $F \le \mu G$.
In general the equilibrium is possible only for $F \le \min\left(\mu G, \frac{s}{R} G\right)$.

Example 2. The motor wheel.
A wheel of weight $\overline{G}$ and radius R is pulled by the horizontal force $\overline{F}$ and a motor couple $\overline{M}_m$ is acting at the wheel axis. The coefficients of sliding friction and rolling friction are respectively $\mu$ and s. Determine the magnitude of the applied force $\overline{F}$ at equilibrium (Fig. 7.14).

The simple support in A is replaced A by: a normal reaction $\overline{N}$, the sliding friction $\overline{T}$ opposite to the sliding tendency and the resistant couple of the rolling friction $\overline{M}_r$, for which there are two possible senses.

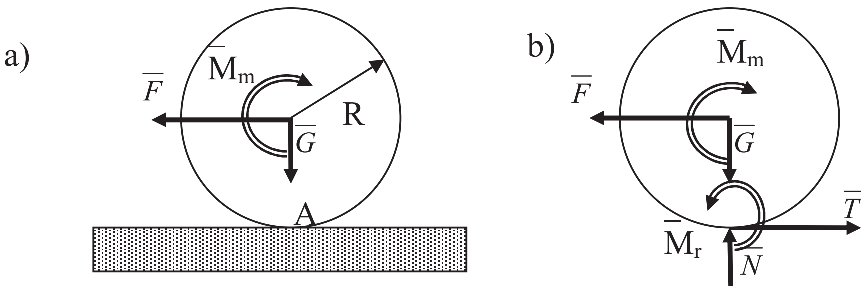



**Fig. 7.14 The equilibrium of a motor wheel. Acting forces and motor moment (a), free body diagram (b)**

It is assumed here that the motor couple prevails and the tendency of rolling corresponds to its action.
The conditions for equilibrium are:

$$\sum_{i=1}^{2} X_i = 0: \ T - F = 0$$

$$\sum_{i=1}^{2} Y_i = 0: \ N - G = 0$$

$$\sum_{i=1}^{3} M_{io} = 0: \ M_r + FR - M_m = 0$$

and the inequalities, valid in conditions of equilibrium: $T \le \mu N; \ M_r \le sN$.
From the first three equations it follows that $F \le \mu G; \ M_m - FR \le sG$.

In general the equilibrium is possible for $\frac{M_m - sG}{R} \le F \le \mu G$.

If the force F is below the minimum value, the wheel begins to roll under the action of the motor couple. If on the contrary, it surpasses the maximum value, the wheel will also slide in the sense of the applied force.

### 7.5.3. The pivoting friction

The pivoting friction is the friction occurring during the relative rotation of two bodies, about a common normal to their surfaces in contact, or if such a tendency of motion exists (Fig. 7.15). As the pivoting motion is a rotation about an axis, the pivoting friction may be replaced by a resistant couple, whose moment vector $\bar{M}_p$ acts along the common normal to these surfaces and whose sense is inverse to the pivoting tendency of motion of the given body (I) with respect to the body (II).

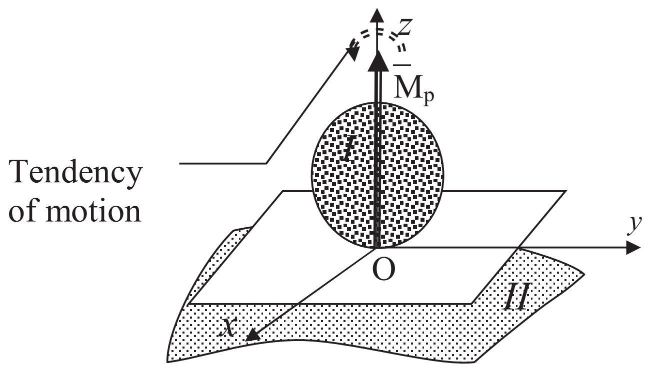



**Fig. 7.15. Pivoting friction moment and tendency of motion**

The magnitude $|\bar{M}_p|$ of the **pivoting moment** is limited by:

$$\left|\bar{M}_p\right| \le \mu_p \left|\bar{N}\right|. \qquad (7.15)$$

In some technical cases, a vertical shaft is supported by a pivoting friction bearing. It is possible to obtain and explicit expression of the **pivoting friction coefficient**

$\mu_p$ as a function of the coefficient of the sliding friction $\mu$ and of the radius $R$ of the bearing if we adopt certain hypotheses, namely if the pressure $p$ and the static coefficient of the sliding friction are considered to be constant on the surface of contact between the shaft and the bearing (Fig. 7.16). Using these hypotheses it follows that $p = \dfrac{N}{\pi R^2}$ so that

$$\left|\overline{M}_p\right|_{\max} = \int_S r\mu p dS = \iint r\mu \frac{N}{\pi R^2} r dr d\theta = \frac{\mu N}{\pi R^2}\int_0^{2\pi} d\theta \int_0^R r^2 dr$$
$$= \frac{\mu N}{\pi R^2}\frac{2\pi R^3}{3} = \frac{2}{3}\mu N R \qquad \Rightarrow \qquad \mu_p = \frac{2}{3}\mu R \tag{7.16}$$

and (7.15) becomes

$$\left|\overline{M}_p\right| \le \frac{2}{3}\mu R\left|\overline{N}\right|. \tag{7.17}$$

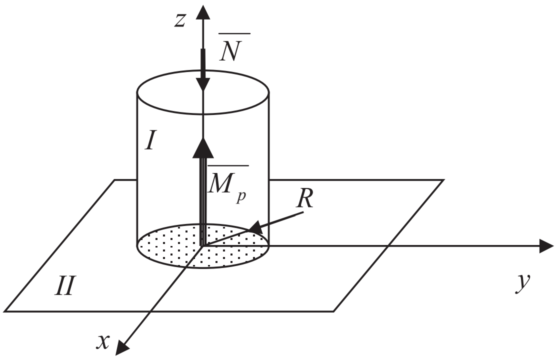
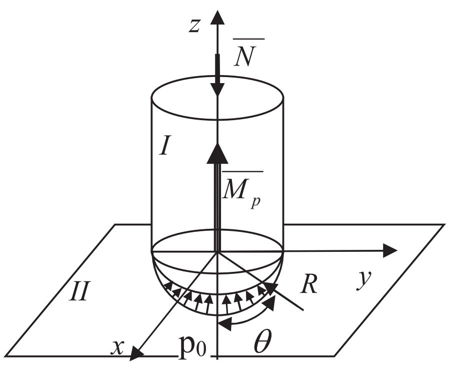



**Fig. 7.16. Pivoting friction for uniform pressure on a cylinder with plane circular cap**




Example.
A vertical shaft has a hemispherical cap of radius R and the distributed normal reaction force is defined by $p(\theta) = p_0 \cos\theta$, in which $\theta$ is the angle to the shaft axis (Fig. 7.17). Find the pivoting friction coefficient $\mu_p$ as a function of the coefficient of the sliding friction $\mu$.

**Fig. 7.17 Pivoting friction of a shaft with hemispherical cap**

The distributed force produces an elementary vertical force, which is a projection on the Oz axis of the elementary normal force:

$dN = pdS\cos\theta = \left(p_0\cos\theta\right)\left(2\pi R\sin\theta\cdot Rd\theta\right)\cos\theta = \pi R^2 p_0\left(2\cos^2\theta\sin\theta d\theta\right)$;

$\theta\in\left[0,\frac{\pi}{2}\right]$.Consequently $N = \pi R^2 p_0\left(2\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta\right)$ and substituting $\cos\theta = u$, the integral becomes $N = 2\pi R^2 p_0\int_0^1 u^2 du = \frac{2}{3}\pi R^2 p_0$, from which $p_0 = \frac{3}{2}\frac{N}{\pi R^2}$.

On the other hand the elementary pivoting moment is

$dM_p = \mu\pi R^2 p_0\left(2\cos^2\theta\sin\theta d\theta\right)R\sin\theta = \frac{\mu}{2}\pi R^3 p_0\left(\sin^2 2\theta\right)d\theta$, so that

$\left|M_p\right|_{\max} = \mu\frac{\pi^2}{8}R^3 p_0 = \mu\frac{3\pi}{16}RN$, from which $\mu_p = \frac{3\pi}{16}\mu R \simeq 0.589\mu R < \frac{2}{3}\mu R$. The pivoting friction coefficient is smaller for the same materials and radius than the cylindrical pivot with plane circular cap.

### 7.5.4. The friction between shaft and bearing

The relative movement of a shaft with respect to a journal bearing is a rotation. The friction between a shaft and the bearing may be replaced by a moment vector $\overline{M_f}$ acting along the axis of the shaft and whose sense is opposite to the possible rotation (Fig. 7.18).

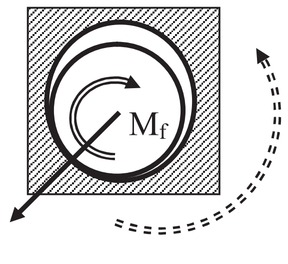



**Fig. 7.18 Friction between the shaft and the journal bearing.**

The magnitude of this moment is limited to:

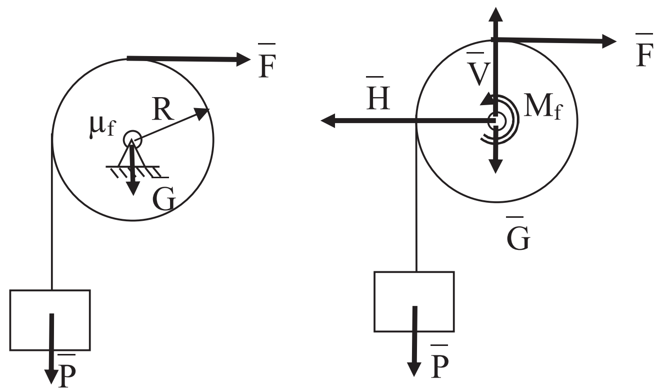
$$\left|\overline{M_f}\right| \le \mu_f r\left|\overline{R}\right| + \left|\overline{M_{f0}}\right|, \tag{7.18}$$

where $\mu_f$ is the coefficient of friction in the shaft, $r$ is the radius of the shaft, $\left|\overline{R}\right| = \sqrt{H^2+V^2}$ is the magnitude of the reaction force in the journal bearing and $\overline{M}_{f0}$ is the moment of friction if the bearing is tightened. If the bearing is not tightened (7.18) becomes:

$$\left|\overline{M_f}\right| \le \mu_f r\left|\overline{R}\right|. \tag{7.19}$$

Example

A fixed pulley of radius R and weight G is used to maintain in equilibrium a weight P with an applied force F. The axle of the pulley has a friction coefficient $\mu_f$. Find the magnitude of the applied force at equilibrium (Fig. 7.19).




The hinge is replaced by the vertical V and horizontal H components of the reaction force. The friction couple $M_f$ is assumed to be as indicated in the figure, but the opposite sense is also possible since there is no unique tendency of motion.

**Fig. 7.19 A pulley with friction in the bearing**

The equilibrium conditions are:

$$\sum_{i=1}^{2} X_i = 0: \; F - H = 0$$

$$\sum_{i=1}^{2} Y_i = 0: \; V - G - P = 0$$

$$\sum_{i=1}^{3} M_{io} = 0: \; PR + M_f - FR = 0$$

and the friction couple at equilibrium is limited to $M_f \le \mu_f \sqrt{H^2 + V^2}$.

Replacing in the inequality the forces from the equilibrium equations, it can be written:

$$F - P \le \frac{\mu_f}{R}\sqrt{F^2 + (G+P)^2} \text{ or } F^2\left[1 - \left(\frac{\mu_f}{R}\right)^2\right] - 2FP - \left(\frac{\mu_f}{R}\right)^2 \left(G^2 + 2GP\right) \le 0$$

The mathematical solution is represented by forces F between the roots of the associated algebraic equation:

$$F = \frac{P \pm \sqrt{P^2 + \left[1-\beta^2\right]\beta^2\left(G^2 + 2GP\right)}}{1-\beta^2} \text{ in which } \beta = \frac{\mu_f}{R}.$$

Since one of the roots is negative and a cable can be only tensioned, the effective solution is

$$0 \le F \le \frac{P + \sqrt{P^2 + \left[1-\beta^2\right]\beta^2\left(G^2 + 2GP\right)}}{1-\beta^2}.$$

For $\mu_f = 0$, the unique solution $F=P$ remains possible and the friction moment vanishes.

# 8. STATICS OF SYSTEMS OF MATERIAL POINTS AND RIGID BODIES

## 8.1. Conditions of equilibrium

A system of $n$ material points $A_i, i = 1,2,...,n$ is considered. Denoted as $\overline{F}_i$, $i = 1,...,n$, are the **external forces** acting upon these material points (the forces produced by interactions with material points or bodies not belonging to the considered system). The forces $\overline{F}_{ij}\left(i, j = 1,2,...,n; i \neq j\right)$ are called **internal forces** and occur in pairs due to the principle of action and reaction: $\overline{F}_{ij}$ acts on the material point $A_i$, being oriented towards the point $A_j$ and $\overline{F}_{ij} = -\overline{F}_{ji}$, in which $\overline{F}_{ji}$ is the force acting on the material point $A_j$. It follows that:

$$\overline{F}_{ij} + \overline{F}_{ji} = \overline{0}; \qquad \overline{r}_i \times \overline{F}_{ij} + \overline{r}_i \times \overline{F}_{ji} = \overline{0}. \tag{8.1}$$

The system of forces acting on a system of material points is in equilibrium if and only if the forces acting on each material point are in equilibrium.

It follows the necessary and sufficient conditions:

$$\overline{F}_i + \sum_{j=1}^{n} \overline{F}_{ij} = \overline{0}; \; i = 1...n, \tag{8.2}$$

written using the convention $\overline{F}_{ii} \equiv 0; \; \forall i = 1...n$.

It is possible to obtain necessary equilibrium conditions for the external forces $\overline{F}_i$ by eliminating the internal forces in (8.2):

a) By adding these equations;

b) By vector multiplication of these equations by $\overline{r}_i$ and adding afterwards the resulting equations. The results are:

$$\sum_{i=1}^{n} \overline{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} = \overline{0}; \quad \sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{r}_i \times \overline{F}_{ij} = \overline{0} \tag{8.3}$$

The relations (8.1) can be rewritten by summation:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} = \overline{0}; \quad \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{r}_i \times \overline{F}_{ij} = \overline{0} \tag{8.4}$$

and (8.3) becomes:

$$\sum_{i=1}^{n} \overline{F}_i = \overline{0}; \quad \sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i = \overline{0} \tag{8.5}$$

Note that the conditions (8.5) for external forces look the same as the necessary and sufficient conditions for the equilibrium of a system of forces acting on a rigid body, but these conditions are only necessary for the equilibrium of a

system of material points. These conclusions are also valid for a system of rigid bodies.
A system of forces acting on a system of rigid bodies is in equilibrium if and only if the forces acting on each rigid body are in equilibrium.
The conditions (8.5) are only necessary but not sufficient for the equilibrium of a system of rigid bodies. The conditions (8.5) express the Principle of Rigidity:
If a system of material points or rigid bodies is in equilibrium, then the system of external forces is equivalent to zero.
This is valid for any system which may be a part of a rigid body, or a piece of an elastic material, or even a volume of fluid.

## *8.2. Systems of trusses. Frames*

### 8.2.1. Mobility analysis

A system of perfectly rigid bars (trusses), pin-connected and forming as a whole a rigid body is called a frame. In Fig. 8.1 is shown a simple frame (or truss), as used in bridges. It consists of steel gird riveted together at the joints.

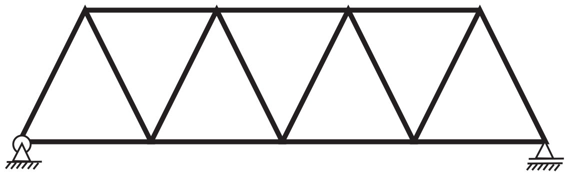


**Fig. 8.1 A system of trusses**

For the mathematical approach the system is approximated as follows:
a) The girders are treated as weightless rigid bars (trusses);
b) The joints are supposed to be smoothly working hinges. Each bar is in principle free to rotate about the pin joints without any resisting couple, but in general the geometrical configuration will not allow it. The joints of a frame are also called **nodes**.

In the following a plane frame is considered. Taking arbitrary axes in the plane of the frame, the coordinates of the nodes are denoted by $(x_1, y_1)$, $(x_2, y_2)$,...,$(x_n, y_n)$. There are 2n coordinates altogether.
If the nodes $i$ and $j$ are connected by a bar of length $l_{jj}$, their coordinates must satisfy the relation

$$\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 = l_{ij}^2 \tag{8.6}$$

Thus, if there are $b$ bars, the $2n$ coordinates are subjected to $b$ relations of this type. If the plane frame has $h$ hinges ($h$ fixed nodes) and $s$ simple supports ($s$ nodes are constrained to move on $s$ lines), there are $2h+s$, restricted displacements conditions. Supposing all these conditions to be independent, if the plane frame is a just rigid one (i.e the removal of one of its bars destroys its rigidity) these $b+2h+s$

independent conditions suffice to determine the whole frame. The *2n* coordinates of the nodes can be determined. Hence

$$b + 2h + s = 2n \tag{8.7}$$

Frequently (Fig. 8.1), *h=l*, *s=1* and (8.7) becomes

$$b + 3 = 2n \tag{8.8}$$

Such a just-rigid plane frame is shown in the mentioned in Fig. 8.2a. If *b + 2h + s < 2n*, the plane frame is not rigid; it becomes a **mechanism** as is shown in Fig. 8.2a, because *b + 2h + s = 4 + 2·1+1 = 7* and *2n = 2 x 4 = 8*). If *b + 2h + s > 2n* the plane frame has redundant bars i.e. the removal of such a bar will not affect the rigidity of the plane frame, as it is shown in Fig. 8.2b (*b + 2h + s = 6 + 2·1+1 = 9*; *2n = 2·4 = 8*).

The relations (8.7) and (8.8) are only necessary but not sufficient. Indeed, the system of bars shown in Fig. 8.2c satisfies the relation (8.7) (*b + 2h + s = 9 + 2· 1 + 1 = 12; 2n = 2· 6 = 12*), but it is not a just-rigid plane frame. In fact it is a mechanism with a redundant bar.

Also, the system of bars shown in Fig. 8.2d satisfies the relation (8.7) (*b + 2h + s = 2 + 2·2+ 0 = 6; 2n = 2·3 = 6*), but it is a **critical system** (it has infinitesimal displacements indicated by the dotted lines), because the *b + 2h + s* conditions are not independent in this case

For example the equations:

$$\begin{aligned} (x_1 - x_2)^2 + (y_1 - y_2)^2 &= l_{12}^2 \\ (x_3 - x_2)^2 + (y_3 - y_2)^2 &= l_{23}^2 \\ (x_3 - x_1)^2 + (y_3 - y_1)^2 &= l_{31}^2 \end{aligned} \tag{8.9}$$

are not independent if $l_{13} = l_{12} + l_{23}$.

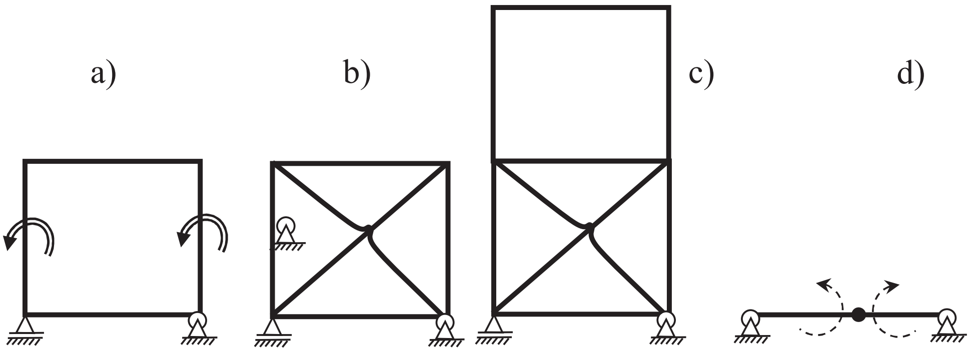



**Fig. 8.2 Frames. Mechanism (a), redundant (b), mechanism & redundant (c), critical (d).**

Similarly for a space frame the following condition necessary but not sufficient for its just-rigidity can be obtained

$$b + 3a + s = 3n \tag{8.10}$$

in which *b* is the number of bars, *n* is the number of nodes, *a* the number of spherical articulation joints and *s* the number of simple supports.

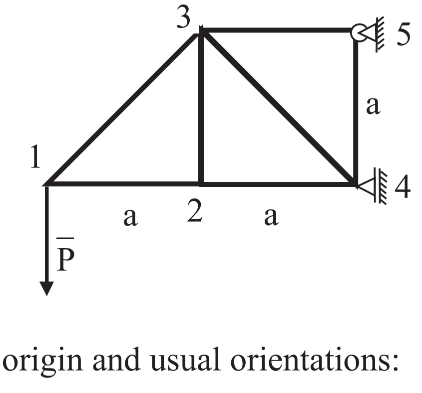
### 8.2.2. Tensions in trusses

A just-rigid frame is considered to be fixed by some external constraints so that it cannot move as a rigid body. Consider that external forces (loads) are applied to some or all the nodes. Each bar is in equilibrium under the action of the two forces at its ends. These two forces must be equal in magnitude and act in opposite senses along the bar, at equilibrium. If the two forces are stretching the bar, the bar is said to be in **tension**. If the forces are pressing the bar, the bar is said to be in **compression**. The word tension is used to cover both cases. A plus sign is associated with tension and a minus sign with compression. For a just-rigid frame in equilibrium, two problems arise: a) Determine the external reactions at the supporting joints; b) Determine the tensions in the trusses.

The first problem may be solved if the principle of rigidity is applied. For the second problem some other methods may be used: analytical method of nodes, analytical method of sections and others.

### 8.2.3. Analytical method of nodes

Each node may be considered as a particle in equilibrium, under the action of an external force and the reactions of the bars meeting in that point. If the frame is a planar one, all these forces lie in a plane. There are two scalar equations of equilibrium for each node and thus $2n$ equations in all, if the number of nodes is $n$. These equations involve $2h + s$ unknown components of external reactions and $b$ unknown stresses in bars ($h$ is the number of hinges, $s$ the number of simple supports and $b$ is the number of bars). Thus the total number of unknowns is $2h + s + b$ and a number of $2n$ linear equations to find them. Thus, in a just-rigid frame the problem of finding the external reactions at the supports and hinges and the stresses in the bars is a determinate problem, because as shown in the previous paragraph: $2h + s + b = 2n$.




Example.

The frame in the figure has a vertical force P acting as shown. The length of vertical and horizontal trusses is $a$. Find the tensions in all trusses.

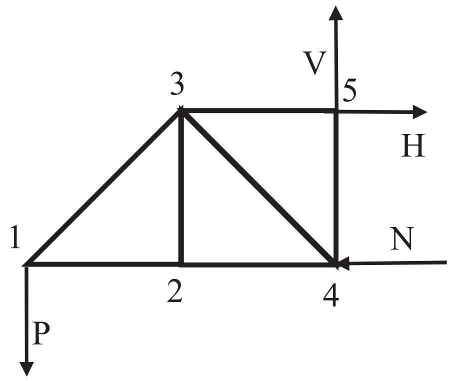
Using the rigidity principle, the equations of equilibrium for the frame can be written with respect to a coordinate frame with point 5 as origin and usual orientations:

$$\sum_{i=1}^{2} X_i = 0: \; H - N = 0$$

$$\sum_{i=1}^{2} Y_i = 0: \; V - P = 0$$

$$\sum_{i=1}^{2} M_{io} = 0: \; 2aP - aN = 0$$

so that *N=H=2P; V=P*.




The forces acting on each node are assumed to be pulling the node, which corresponds to positive tension in the rods.
The equilibrium conditions for node 1 are:

$$\sum_{i=1}^{3} X_i = 0: \; T_{12} + T_{13}\cos\frac{\pi}{4} = 0$$
$$\sum_{i=1}^{3} Y_i = 0: \; T_{13}\sin\frac{\pi}{4} - P = 0 .$$

It follows that $T_{13} = \sqrt{2}P; \; T_{12} = -P$, i.e. a tension and a compression.
For node 2 equilibrium equations are:

$$\sum_{i=1}^{3} X_i = 0: \; -T_{12} + T_{24} = 0$$
$$\sum_{i=1}^{3} Y_i = 0: \; T_{23} = 0 .$$

It follows that both 2-4 and 1-2 trusses are compressed: $T_{24} = T_{12} = -P$ and the rod 2-3 is apparently useless since no force is acting on it. In fact considerations of elastic stability, which are beyond the scope of this work, recommend its presence.
For node 4 equilibrium equations are

$$\sum_{i=1}^{3} X_i = 0: \; -T_{24} - N - T_{34}\cos\frac{\pi}{4} = 0$$
$$\sum_{i=1}^{3} Y_i = 0: \; T_{34}\sin\frac{\pi}{4} + T_{45} = 0 .$$

It follows a compression force for the truss 3-4 and a tension force for truss 4-5: $T_{34} = -\sqrt{2}P; \; T_{45} = P$.
For node 5 equilibrium equations are:

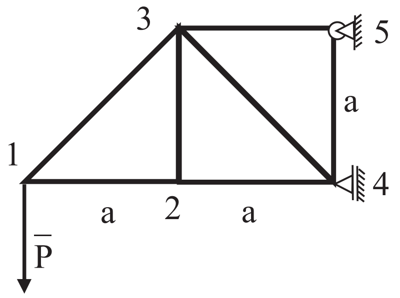
$$\sum_{i=1}^{4} X_i = 0: \; -T_{35} + H = 0$$
$$\sum_{i=1}^{4} Y_i = 0: \; -T_{45} + V = 0 .$$

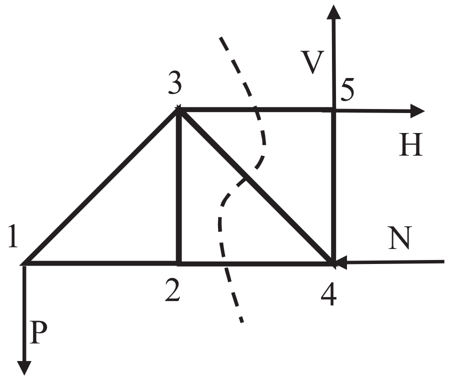
It follows $T_{35} = 2P$ and the last equation can be used for checking the value of *V*. Node 3 was not used since the rigidity principle solved already for 3 unknowns.

### 8.2.4. Analytical method of sections

When it is required only the tension in one or some of the bars, the method of nodes may prove unnecessarily laborious. It is assumed that the plane frame is such that it is possible to divide it in two by cutting only three trusses which are not crossing in a point. If at least two of the bars to cut are not parallel, then it is possible to determine the tensions in the bars without calculating the stresses in the remaining bars. The principle of rigidity is applied on the whole frame and again on one of the two parts.

Example.




The frame in the figure has a vertical force P acting as shown. The length of vertical and horizontal trusses is *a*. Find the tension in the truss 3-4.

Answer.

Using the rigidity principle, the equations of equilibrium for the frame can be written with respect to a coordinate frame with point 5 as origin and usual orientations. The solution determined in the previous example can be used: *N=H=2P; V=P*.

The frame is "cut" as indicated by the dotted line.

Since the rods 3-5 and 2-4 are horizontal, it is recommended to write the equation of equilibrium on the vertical direction for the left part of the frame: $\sum_{i=1}^{2} Y_i = 0$: $-T_{34} \sin\frac{\pi}{4} - P = 0$, from which $T_{34} = -\sqrt{2}P$, the same as in the previous example but writing a single equation.

Note. These methods are useful only for perfectly rigid rods, hinged at their respective ends. If elasticity is to be taken into account, other methods should be used.

# 9. FLEXIBLE CABLES

## 9.1. Basic equations

A flexible cable is different from a rod basically because it can be easily deformed. In mechanics, this property is idealized and a flexible cable is considered a material curvilinear line, having no bending rigidity. The word “cable” used in the following can represent practical applications such as chains, ropes, strings and threads. The theoretical predictions agree well with the results of experiments conducted on cables in which the bending rigidity is small.

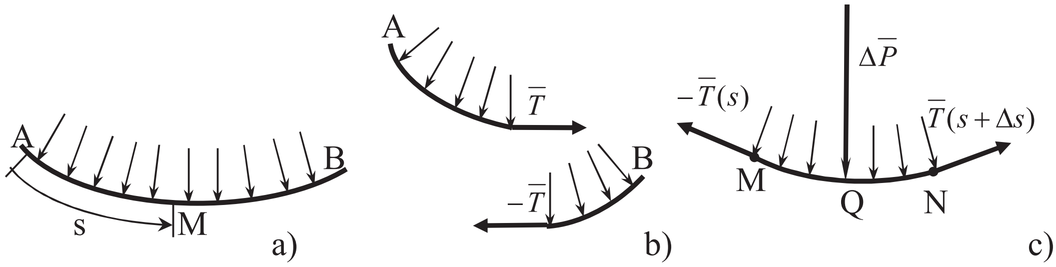



**Fig. 9.1 Elastic cable. Positioning (a) . Section (b). Elementary arc (c).**

A flexible cable is at rest under the action of known external forces and the tension at its ands A and B (Fig. 9.1). The notation *s=AM* represents the curvilinear coordinate of an arbitrary point M on the material line.

If the bending rigidity of the flexible cable is negligible, then any cross section of the cable can be considered to be a hinge.

If the cable is cut in a point M and the right part is removed, then the left part AM will remain at rest after the addition of the tension $\overline{T} = \overline{T}(s)$ (Fig. 9.1b). Fig. 9.1c shows a small arc MN=$\Delta s$ of the cable. The forces acting on the arc MN are the tensions $-\overline{T}(s)$, $\overline{T}(s+\Delta s)$ and the external force $\Delta\overline{P}$. The element MN is at rest under the action of these forces, and so the principle of rigidity can be applied:

$$-\overline{T}(s) + \overline{T}(s+\Delta s) + \overline{P} = \overline{0} \tag{9.1}$$

If this equation is divided by $\Delta s$ and if $\Delta s \to 0$, one gets:

$$\begin{gathered}\lim_{\Delta s \to 0} \frac{\overline{T}(s+\Delta s) - \overline{T}(s)}{\Delta s} + \lim_{\Delta s \to 0} \frac{\Delta\overline{P}}{\Delta s} = \overline{0} \\ \lim_{\Delta s \to 0} \frac{\overline{\mathrm{MN}}}{\Delta s} \times \overline{T}(s+ds) + \lim_{\Delta s \to 0} \overline{\mathrm{MQ}} \times \frac{\Delta\overline{P}}{\Delta s} = \overline{0}\end{gathered} \tag{9.2}$$

It follows that:

$$\frac{d\overline{T}}{ds} + \overline{p} = \overline{0}; \quad \overline{\tau} \times \overline{T} = \overline{0}. \tag{9.3}$$

The following expressions have been used:

$$\lim_{\Delta s \to 0} \frac{\overline{T}(s+\Delta s) - \overline{T}(s)}{\Delta s} = \frac{d\overline{T}}{ds} \tag{9.4}$$

$$\lim_{\Delta s \to 0} \frac{\Delta \overline{P}}{\Delta s} = \overline{p};\ \lim_{\Delta s \to 0} \frac{\overline{\mathrm{MN}}}{\Delta s} = \frac{d\overline{r}}{ds} = \overline{\tau};\ \lim_{\Delta s \to 0} \overline{\mathrm{M}Q} = \overline{0} \tag{9.5}$$

where $\overline{p}$ is the external force per unit length of cable and $\overline{\tau}$ is the unit tangent vector at M. Form the formula $\overline{\tau} \times \overline{T} = 0$ it follows the colinearity condition:

$$\overline{T} = T \cdot \overline{\tau} \tag{9.6}$$

Note that always $T$>0 because no compression can exist in a flexible cable. The first equation (9.3) becomes:

$$\frac{d}{ds}(T \cdot \overline{\tau}) + \overline{p} = \overline{0}. \tag{9.7}$$

Using the Cartesian coordinates, the vector equation becomes

$$\begin{cases} \frac{d}{ds}(T \cdot \frac{dx}{ds}) + p_x = 0 \\ \frac{d}{ds}(T \cdot \frac{dy}{ds}) + p_y = 0 \\ \frac{d}{ds}(T \cdot \frac{dz}{ds}) + p_z = 0 \end{cases} \tag{9.8}$$

because

$$\overline{\tau} = \frac{d\overline{r}}{ds} = \frac{dx}{ds} \cdot \overline{i} + \frac{dy}{ds} \cdot \overline{j} + \frac{dz}{ds} \cdot \overline{k} \tag{9.9}$$

Using the first of the Frénet-Serret formulas:

$$\frac{d\overline{\tau}}{ds} = \frac{1}{\rho} \cdot \overline{\nu} \tag{9.10}$$

where $\rho$ is the radius of curvature at M and $\overline{\nu}$ is the unit principal normal vector (the plane of $\overline{\tau}$ and $\overline{\nu}$ is called the osculating plane). The formula (9.7) can be cast in the form:

$$\frac{dT}{ds} \cdot \overline{\tau} + T \cdot \frac{1}{\rho} \cdot \overline{\nu} + \overline{p} = \overline{0}. \tag{9.11}$$

From this vector formula, the following scalar equations can be obtained:

$$\frac{dT}{ds} + p_\tau = 0;\ \frac{T}{\rho} + p_\nu = 0;\ p_\beta = 0 \tag{9.12}$$

The last of these equations expresses the fact that the osculating plane at each point contains the external force vector $\overline{p}$.

### 9.1.1. Flexible cable without transverse external forces

In this case, $\overline{p} = \overline{0}$ and (9.12) becomes:

$$\frac{dT}{ds} = 0; \; \frac{T}{\rho} = 0; \; 0 = 0 \tag{9.13}$$

It follows that the tension $T$ in the cable is constant. If $T \neq 0$, $\frac{1}{\rho} = 0$. Therefore, the flexible cable has a straight line form. If however $T$=0, the curve is arbitrary.

### 9.1.2. Flexible cable on a smooth surface without applied forces

Since the reaction is normal to the surface, it is also normal to the curve, and so $p_\tau = 0$. But $p_\beta = 0$ according to the last equation (9.12). It follows that $\overline{p} = p_\upsilon \cdot \overline{\nu}$ since the principal normal unit vector $\overline{\nu}$ of the curve is normal to the surface at each point. It follows that the curve is a geodesic of the surface. Since $p_\tau = 0$, $\frac{dT}{ds} = 0$ and it follows that the scalar $T$ of the tension is constant; in particular the tensions at the cable ends are equal.

### 9.1.3. Flexible cable in contact with a rough surface

In this case each elementary arc of the cable is subjected to the action of a normal elementary force N and an elementary friction force Φ (Fig. 9.2):

$$p_\nu = -N; \; p_\tau = -\Phi; \; -\mu N < \Phi < \mu N \tag{9.14}$$

where $\mu$ is the coefficient of Coulomb for the sliding friction.

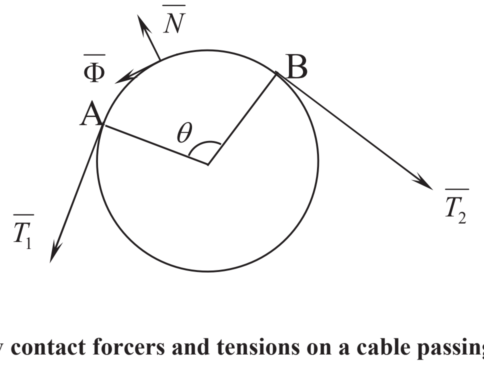



**Fig. 9.2 Elementary contact forcers and tensions on a cable passing over a rough surface**

The equation (9.12) becomes:

$$\frac{dT}{ds} - \Phi = 0; \quad \frac{T}{\rho} - N = 0; \quad -\mu \cdot \frac{T}{\rho} \le \frac{dT}{ds} \le \mu \cdot \frac{T}{\rho} \tag{9.15}$$

Since the elementary arc has a length $ds = \rho \cdot d\theta$, $\rho > 0$ and $T>0$, the last inequality can be written:

$$-\mu \cdot d\theta \le \frac{dT}{T} \le \mu \cdot d\theta \tag{9.16}$$

and by integrating along the arc of contact θ, the inequality becomes:

$$-\mu \cdot \theta \le \ln\frac{T}{T_0} \le \mu \cdot \theta \tag{9.17}$$

or:

$$T_0 \cdot e^{-\mu\theta} \le T \le T_0 \cdot e^{\mu\theta} \tag{9.18}$$

The rapid increase of the exponential with increasing $\theta$ is of great practical importance.

<u>Example</u>
Consider a rope wrapped twice around a post, for which the coefficient of friction is $\mu = 0.5$. Then $T = T_0 \cdot e^{0.5 \cdot 4\pi} = e^{2\pi} \cdot T_0 \cong 553.5 \cdot T_0$
Thus, a load $T$ can be sustained by application of a force $T_0$ of less than $T/500$. A much greater load might be sustained if the rope were wrapped more than 10 times around the post. This principle is used to holding ships by ropes passed by mooring posts and in hoists in which a rope is passed round a revolving drum, the end being held in the hand!

### 9.1.4. Uniform heavy flexible cable hanging freely

If the weight of flexible cable per unit length $\overline{p} = \text{constant}$, the equations (9.8) become (Fig. 9.3):

$$\begin{cases} \frac{d}{ds}(T \cdot \frac{dx}{ds}) = 0 \\ \frac{d}{ds}(T \cdot \frac{dy}{ds}) - p = 0 \\ \frac{d}{ds}(T \cdot \frac{dz}{ds}) = 0 \end{cases} \tag{9.19}$$

It has been assumed that the two ends of the cable lie in the vertical plane Oxy.
It follows that:

$$T \cdot \frac{dx}{ds} = H; \; T \cdot \frac{dz}{ds} = C \tag{9.20}$$

From the first equation, the tension in the cable can be expressed as :

$$T = H \cdot \frac{ds}{dx} \tag{9.21}$$

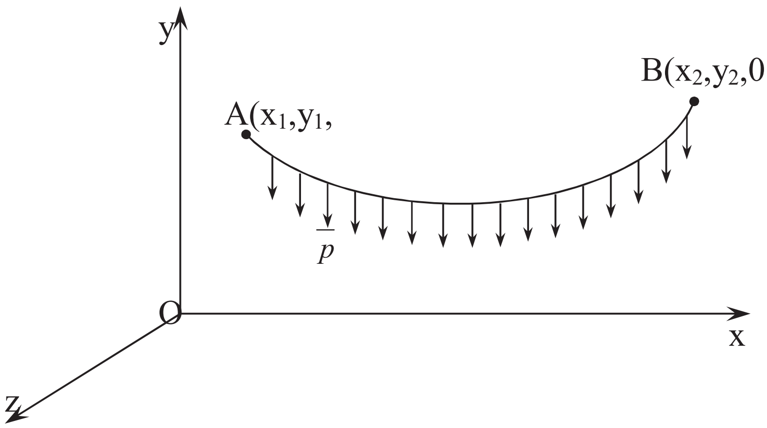



**Fig. 9.3 Cable submitted to a uniform load**

Replacing the last result in the second equation(9.20), it can be found:

$$\frac{dz}{dx} = C_1 \Rightarrow z = C_1 x + C_2 \tag{9.22}$$

As supposed, the ends A and B of the flexible cable are situated in the Ox*y* plane $(z_A = z_B = 0, x_A \neq x_B)$, it follows that:

$$0 = C_1 \cdot x_A + C_2; \quad 0 = C_1 \cdot x_B + C_2 \Rightarrow \; C_1 = C_2 = 0; \;\; z \equiv 0 \tag{9.23}$$

Therefore, the curve is situated in the Ox*y* plane.
The second equation (9.19) becomes:

$$\frac{d}{ds}(H \cdot \frac{dy}{dx}) = p \tag{9.24}$$

Since *H*=constant and $ds = \sqrt{1+(y')^2}dx$ , this equation may be written:

$$\frac{d(y')}{\sqrt{1+(y')^2}} = \frac{p}{H}dx \tag{9.25}$$

Changing variable $y' = \sinh(u) \Rightarrow dy' = \cosh(u)du$ and it also follows for the denominator $\sqrt{1+(y')^2} = \sqrt{1+(\sinh u)^2} = \cosh u$ , so that the last formula becomes $du = \frac{p}{H}dx$.

By integrating again this last formula, it can be obtained:

$$u=\arg(\sinh y') = \frac{x}{a} + \mathrm{A}, \tag{9.26}$$

in which A is an arbitrary constant and

$$a = \frac{H}{p} \tag{9.27}$$

The relation (9.26) may be written:

$$y' = \sinh(\frac{x}{a} + \mathrm{A}) \tag{9.28}$$

By integrating again, it can be obtained:

$$y = a \cdot \cosh\left(\frac{x}{a} + \mathrm{A}\right) + \mathrm{B} \tag{9.29}$$

in which B is another arbitrary constant. This curve is called common catenary (Fig. 9.4).

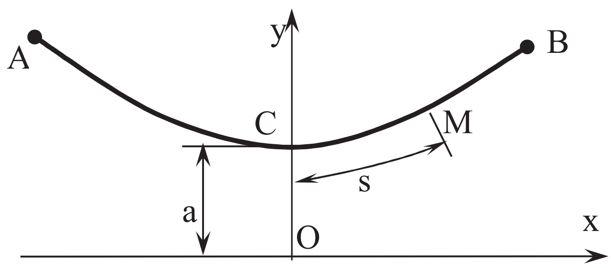



**Fig. 9.4 The common catenary**

If the axis O*y* is the symmetry axis of the curve and the Oxy system of axis is such that the curve passes through the point (0,*a*), the constants A and B are equal to 0. Then, the equation of common catenary becomes:

$$y = a \cdot \cosh\frac{x}{a} \tag{9.30}$$

The length *s* of the cable measured from vertex to a general point is:

$$s = \int ds = \int_0^x \sqrt{1+(y')^2}\,dx = \int_0^x \sqrt{1+\sinh^2\frac{x}{a}}\,dx = \int_0^x \cosh\frac{x}{a}\,dx = a \cdot \sinh\frac{x}{a} \tag{9.31}$$

The tension *T* in an arbitrary point is:

$$T = H \cdot \frac{ds}{dx} = H \cdot \sqrt{1+(y')^2} = H \cdot \sqrt{1+\sinh^2\frac{x}{a}} = H \cdot \cosh\frac{x}{a} = py \tag{9.32}$$

The slope of the cable in an arbitrary point is:

$$\tan\theta = y' = \sinh\frac{x}{a} \tag{9.33}$$

<u>Example</u>

A uniform cable is free under its own weight $p$=50N/m. The span is 200 m. The length of the cable is 201 m. Determine the maximum tension (the ends A and B of the cable are at the same height: $y_A = y_B$).

Using the previous formulas it can be successively deduced:

$\frac{L}{2} = a\cdot\sinh\frac{100}{a} = 100,5$ $\qquad T_{max} = py = 50\cdot a\cdot\cosh\frac{100}{a}$

The first equation is a transcendental one. Its solution is numerically determined with a $10^{-10}$ precision to be $a$=577.782780 m. It follows that $T_{max}$ =29322.908 N.

### 9.1.5. Approximate formulas for the catenary

The exact formulas presented in the previous paragraph have the major inconvenient of requiring solution of a transcendental equation, for which only numerical methods can be used. The next formulas have been established by one of the authors [7] and have the advantage of precision and simplicity.

a) Suppose the length of the cable $L$, the difference of height denoted by $h$ between the end points A and B, and the horizontal distance $d$ between the two points, are given. The following equations are derived applying (9.30) and (9.31):

$$\begin{cases} L = s_B - s_A = a\sinh\frac{x_B}{a} - a\sinh\frac{x_A}{a} \\ h = y_B - y_A = a\cosh\frac{x_B}{a} - a\cosh\frac{x_A}{a} \end{cases} \tag{9.34}$$

Taking the square of both relations and then subtracting them, after simple operations it is deduced:

$$\frac{L^2 - h^2}{a^2} = -2 + 2\cosh\frac{x_B - x_A}{a} \tag{9.35}$$

Replacing the given value $x_B - x_A = d$, this formula becomes:

$$\frac{L^2 - h^2}{2a^2} = \cosh\left(\frac{d}{a}\right) - 1 \tag{9.36}$$

From the series expansion of the hyperbolic cosine, are kept the first four terms:

$$\cosh\frac{d}{a} = 1 + \frac{1}{2!}\left(\frac{d}{a}\right)^2 + \frac{1}{4!}\left(\frac{d}{a}\right)^4 + \frac{1}{6!}\left(\frac{d}{a}\right)^6 + O\left(\left(\frac{d}{a}\right)^8\right) \tag{9.37}$$

Replacing this result in (9.36), an efficient second order equation is obtained for $a^2$:

$$360a^4\left(L^2-h^2-d^2\right)-30d^4a^2-d^6=0 \tag{9.38}$$

<u>Example</u>
For the same data as in the previous example, find the parameter $a$ and the maximum tension in the cable

The equation (9.38) becomes:
$360a^4\left(201^2-200^2\right)-30\cdot 200^4\cdot a^2-200^6=0$, and a=577.780327 which coincides up to 2 decimal places with the more precise numerical solution, but much easier to find. $T_{max}=py=50\cdot a\cdot\cosh\frac{100}{a}$ = 29322.788N (0.0007% error)

b) Suppose as given the length of the cable $L$, and the horizontal distance $d$ between the two points, the two ends A and B are at the same height. The formula (9.31) relates the length of the cable to the parameter $a$:

$$L=2a\sinh\frac{d}{2a}, \tag{9.39}$$

but the solution is only based on numerical methods. However a simple mathematical formula has be deduced [7] by expanding the hyperbolic cosine in its Taylor series and keeping the first three terms:

$$\sinh\frac{d}{2a}=\frac{d}{2a}+\frac{1}{3!}\left(\frac{d}{2a}\right)^3+\frac{1}{5!}\left(\frac{d}{2a}\right)^5+O\left(\left(\frac{d}{2a}\right)^7\right) \tag{9.40}$$

Replacing this formula in (9.39) , a simple second degree equation in $a^2$ is obtained:

$$\left(L-d\right)a^4-\frac{d^3}{24}a^2-\frac{d^5}{1920}=0 \tag{9.41}$$

<u>Example</u>
For the same data as in the previous example, find the parameter $a$ and the maximum tension in the cable.

The equation (9.41) becomes $a^4-\frac{200^3}{24}a^2-\frac{200^5}{1920}=0$, from which a= 577.782472 which coincides up to three decimal places with the numerical solution 577.782780, being thus even more precise than the previous formula. $T_{max}=py=50\cdot a\cdot\cosh\frac{100}{a}$ = 29322.894N (0.00005% error)

### 9.1.6. Strongly stretched heavy uniform flexible cable hanging freely

In general, flexible cables hanging freely are strongly stretched. In this case, the parameter $a = \frac{H}{p}$ of the common catenary is very large, because the tension $H$ is very high. It follows that $x/a$ is very small. The power-series expansion of $\sinh\frac{x}{a}$ and $\cosh\frac{x}{a}$, can be used so that the previous formulas become:

$$y = a \cdot \cosh\frac{x}{a} = a \cdot \left[1 + \frac{1}{2!}\cdot\frac{x^2}{a^2} + \frac{1}{4!}\cdot\frac{x^4}{a^4} + \ldots\right] = a \cdot \sum_{n=0}^{\infty} \frac{1}{(2n)!}\cdot\left(\frac{x}{a}\right)^{2n}$$

$$s = a \cdot \sinh\frac{x}{a} = a \cdot \left[\frac{x}{a} + \frac{1}{3!}\cdot\frac{x^3}{a^3} + \frac{1}{5!}\cdot\frac{x^5}{a^5} + \ldots\right] = a \cdot \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\cdot\left(\frac{x}{a}\right)^{2n+1} \quad (9.42)$$

$$T = py = pa \cdot \left[1 + \frac{1}{2!}\cdot\frac{x^2}{a^2} + \frac{1}{4!}\cdot\frac{x^4}{a^4} + \ldots\right] = pa \cdot \sum_{n=0}^{\infty} \frac{1}{(2n)!}\cdot\left(\frac{x}{a}\right)^{2n}$$

$$\tan\theta = \sinh\frac{x}{a} = \frac{x}{a} + \frac{1}{3!}\cdot\frac{x^3}{a^3} + \frac{1}{5!}\cdot\frac{x^5}{a^5} + \ldots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\cdot\left(\frac{x}{a}\right)^{2n+1}$$

If the higher powers of $x/a$ are neglected, it can be written:

$$y = a + \frac{1}{2}\cdot\frac{x^2}{a}; \quad s = x + \frac{1}{6}\cdot\frac{x^3}{a^2}; \quad T = p \cdot a; \quad \tan\theta = \frac{x}{a} \quad (9.43)$$

If the same Cartesian coordinates shown in Fig. 9.5 is chosen and if are denoted the span by $d$ and the height of the two ends A and B by $f$, the formulas (9.43) become:

$$y = \frac{x^2}{2a}; \quad a \cong \frac{d^2}{8f}; \quad s = x + \frac{1}{6}\cdot\frac{x^3}{a^2}; \quad L \cong d + \frac{l^3}{24a^2} = d + \frac{8f^2}{3d};$$
$$T = p \cdot \frac{d^2}{8f}; \quad \tan\theta = \frac{8fx}{d^2} \quad (9.44)$$

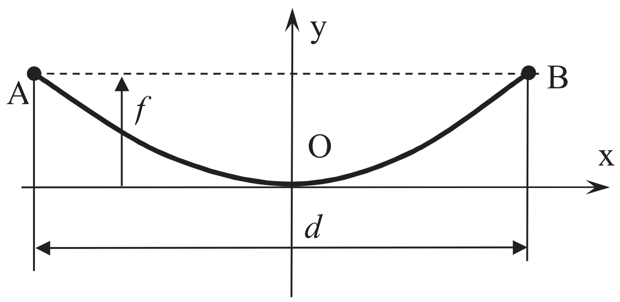



**Fig. 9.5. The Cartesian frame chosen for positioning a strongly stretched cable**

Example

A uniform cable hangs freely under its own weight $p$=50 N/m. The span is $d$=200 m. The length of the cable is $L$=201 m. Determine the maximum tension (the ends A and B of the cable have the same height: $y_A = y_B$).

$$201 = 200 + \frac{8f^2}{3 \cdot 200} \quad ; \quad T = \frac{p \cdot d^2}{8f};$$

The first equation is an algebraic one. Its solution is $f$=8,6602 m. It follows that $T$=28867.69 N. There is a difference between this value and the exact value 29322.908 N (2,6% error), but the formulas are still accepted.

# 10. KINEMATICS OF A POINT

## *10.1. Preliminaries*

Kinematics is part of Mechanics and its objective is studying the motion of particles and rigid bodies, but not the cause of these motions. In kinematics, in addition to Statics concepts, is introduced also the concept of **time**. For the purposes of theoretical kinematics, it is sufficient to assume that at each moment of time there is assigned a certain real number *t*. It is chosen a unit of time, e.g. a second, and an arbitrary staring moment of the motion, which will be called the **initial moment**. The number *t* is positive for moments after and negative for moments before the initial moment.
In kinematics, it is assumed that a certain system of coordinates, called a reference frame is given. The motion of a point or of a body is defined with respect to the **reference frame**. Relative to one frame, a point or a body may be at rest, but relative to another frame those objects may be in motion. A so-called "absolute motion", i.e. the motion of a point or a body independent of other bodies is a meaningless concept. However, in practical problems, a frame attached to certain bodies like the Earth, the Sun, the "fixed" stars etc. can be selected and conventionally the motion with respect to such a frame can be considered as "absolute motion", depending on the required accuracy.

## *10.2. Motion of a point. Path of a moving point. Graph of a motion*

A Cartesian reference frame Oxyz (Fig. 10.1) and a moving point M relative to this reference frame are first considered. The motion of the point can be characterized by means of the position vector $\overline{r} = \overline{OM}$ as a continuous function of the time *t*:

$$\overline{r} = \overline{r}(t). \qquad (10.1)$$

The above vector function (10.1) describes the motion in its entirety, giving at each moment *t* a vector $\overline{r}$ and consequently the position of the point M.
The time can be taken in an interval $t_1 < t < t_2$, which represents an interval of definition for the position vector. The geometric locus of M as time is in the given interval is called path or trajectory of the point.
It can be admitted that the path of the point is the arc Γ (Fig. 10.1b). Along this arc can be chosen certain sense and can be selected an arbitrary point $M_0$, which will be called the origin of arcs. The position of the point M on the arc Γ will be determined by specifying a number *s* whose absolute value is equal to the length of the arc $M_0M$, and which is positive or negative depending on whether the sense of $M_0M$ agrees or not, with the positive selected sense.

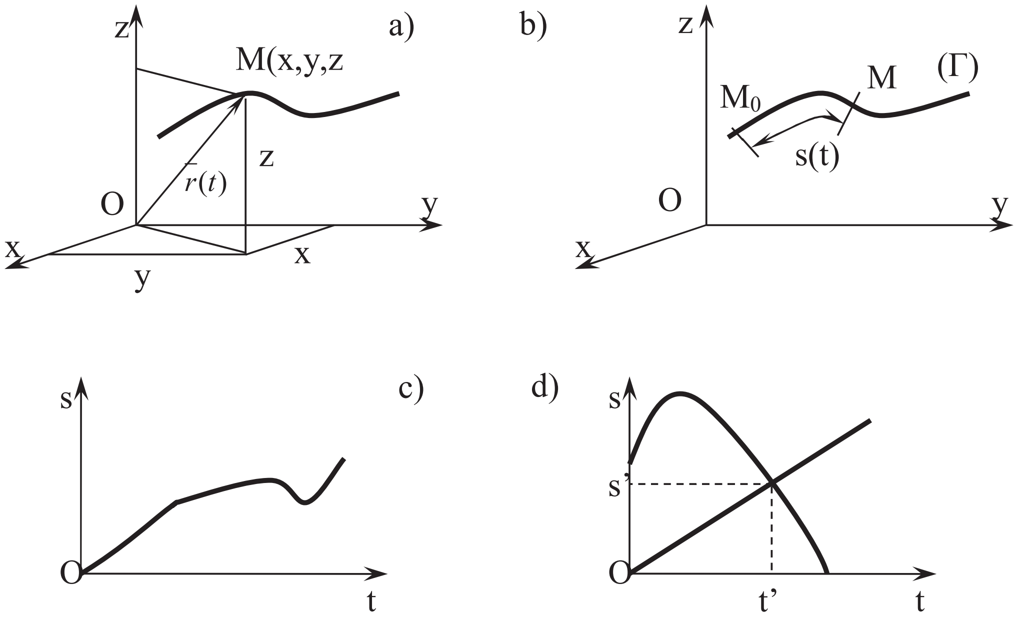



**Fig. 10.1 Reference frame (a), trajectory (b), motion graph (c), two simultaneous motions (d)**

The parameter *s* is called the curvilinear coordinate of the point M on the arc (Γ). The motion of the point M along the arc (Γ) will also be determined by the scalar function:

$$s = s(t). \tag{10.2}$$

By plotting the graph of the function *s(t)* is obtained the motion diagram or the motion graph (Fig. 10.1c). In Fig. 10.1d a example of graph of motions of two particles on the same arc (Γ) is presented. The coordinates (t', s') of the point of intersection of both graphs represent the time and place at which the two particles met.

## *10.3. Velocity*

By definition the velocity of a particle at the point M of position vector $\overline{OM} = \bar{r}(t)$ is the derivative of vector function $\bar{r}(t)$ with respect to time *t*, if this derivative exists:

$$\bar{v} = \frac{d\bar{r}}{dt} = \dot{\bar{r}}. \tag{10.3}$$

The dot above is indicating in what follows, the derivative with respect to time.

It is easy to prove that the velocity vector $\bar{v}$ is tangent to the path of the particle at the point M (Fig. 10.2).

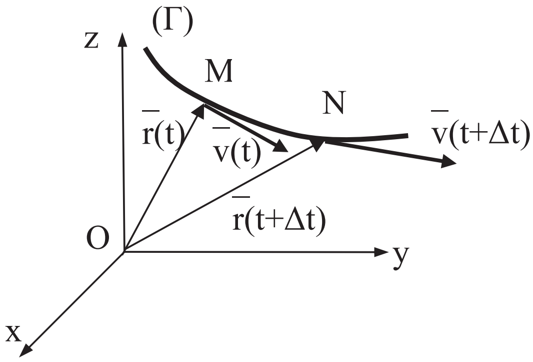



**Fig. 10.2 Two positions M and N along the trajectory of a point**

Suppose that the moving particle is at the point M at a moment of time $t$ and at the point N at the time $t+\Delta t$. It follows that $\overline{OM} = \overline{r}(t)$, $\overline{MN} = \overline{ON} - \overline{OM} = \overline{r}(t+\Delta t) - \overline{r}(t)$, $\overline{ON} = \overline{r}(t+\Delta t)$, so that the velocity vector can be written:

$$\begin{aligned} \overline{v} &= \dot{\overline{r}} = \frac{d\overline{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \overline{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{MN}}{\Delta t} \\ &= \lim_{N \to M} \frac{\overline{MN}}{\left|\overline{MN}\right|} \cdot \lim_{N \to M} \frac{\left|\overline{MN}\right|}{\overset{\frown}{MN}} \cdot \lim_{\Delta t \to 0} \frac{\overset{\frown}{MN}}{\Delta t} = v\overline{\tau} \end{aligned} \qquad (10.4)$$

in which use was made of the known formulas:

$\lim\limits_{N \to M} \dfrac{\overline{MN}}{\left|\overline{MN}\right|} = \overline{\tau}$, which represents the tangent unit vector of the curve (Γ).

$\lim\limits_{N \to M} \dfrac{\left|\overline{MN}\right|}{\overset{\frown}{MN}} = 1$, if the trajectory is a rectifiable curve, which is in general true.

$\lim\limits_{\Delta t \to 0} \dfrac{\overset{\frown}{MN}}{\Delta t} = v$, the scalar of the velocity.

### *10.4. Acceleration*

By definition, the acceleration of a particle at a point M of position vector $\overline{OM} = \overline{r}(t)$ is the first derivative of the vector function $\overline{v}$ with respect to the time $t$, or the second derivative of the vector function $\overline{r}(t)$ with respect to time $t$, if these derivatives exist:

$$\overline{a} = \ddot{\overline{r}} = \dot{\overline{v}}. \qquad (10.5)$$

Suppose that the moving particle is at point M at time $t$ and at point N at time $t+\Delta t$. Let $\overline{v}_M$ and $\overline{v}_N$ be the velocity of the moving particle at the point M and N. Let us apply these two vectors in the same point P (Fig. 10.3). It is easy to remark that:

$$\overline{v}_M = \overline{PM'} = \overline{v}(t); \qquad \overline{v}_N = \overline{PN'} = \overline{v}(t+\Delta t)$$
$$\overline{v}_N - \overline{v}_M = \overline{PN'} - \overline{PM'} = \overline{v}(t+\Delta t) - \overline{v}(t) = \Delta\overline{v} = \overline{M'N'};$$
$$\overline{a} = \frac{d\overline{v}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\overline{v}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\overline{M'N'}}{\Delta t}.$$

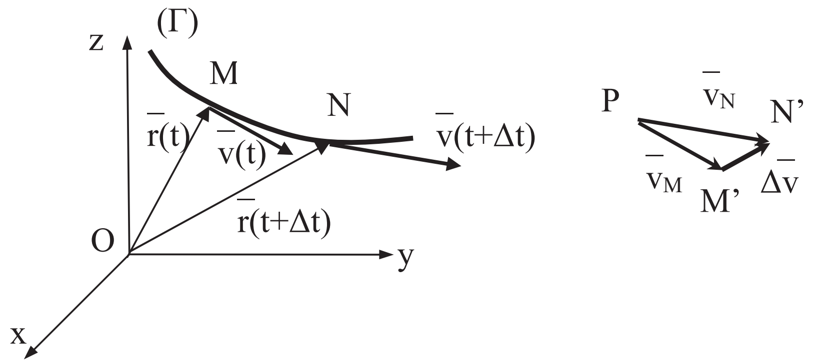



**Fig. 10.3 Acceleration as variation of the velocity vector**

The locus of end points (vertices) of vectors $\overline{v}$ considered applied in an arbitrarily chosen point P is by definition the **hodograph** of the motion. Denote by $\overline{v}_M$, the velocity of the end point $M'$ of vector $\overline{v}_M$ on the holograph at $M'$. It is easy to prove that:

$$\overline{v}_{M'} = \lim_{\Delta t\to 0}\frac{\overline{M'N'}}{\Delta t} = \overline{a}. \qquad (10.6)$$

Therefore the acceleration of a moving point is equal to the "velocity" of the corresponding point on the hodograph of the motion.

### *10.5. Projections of velocity and acceleration on a Cartesian frame*

The position vector $\overline{r}$ of the moving point may be written with respect to the orthogonal axes of a Cartesian coordinates frame, in the form:

$$\overline{r}(t) = x(t)\overline{i} + y(t)\overline{j} + z(t)\overline{k}, \qquad (10.7)$$

where *x(t), y(t), z(t)* are three scalar functions, which are continuous and have derivatives and second order derivatives and $\overline{i}$, $\overline{j}$, $\overline{k}$ are the unit vectors of axes Ox, Oy, Oz respectively. Since the axes are assumed fixed,

$$\dot{\overline{i}} = \overline{0}; \quad \dot{\overline{j}} = \overline{0}; \quad \dot{\overline{k}} = \overline{0} \qquad (10.8)$$

and the expression of the velocity $\overline{v}$ and of the acceleration $\overline{a}$ become:

$$\bar{v}(t) = \dot{\bar{r}}(t) = \dot{x}(t)\bar{i} + \dot{y}(t)\bar{j} + \dot{z}(t)\bar{k}$$
$$\bar{a}(t) = \ddot{\bar{r}}(t) = \ddot{x}(t)\bar{i} + \ddot{y}(t)\bar{j} + \ddot{z}(t)\bar{k} \quad (10.9)$$

It follows that the projections of the velocity and of the acceleration of a moving point on the axes of a Cartesian frame are:

$$v_x = \dot{x}(t); \quad v_y = \dot{y}(t); \quad v_z = \dot{z}(t)$$
$$a_x = \ddot{x}(t); \quad a_y = \ddot{y}(t); \quad a_z = \ddot{z}(t) \quad (10.10)$$

### *10.6. Projections of velocity and acceleration on a cylindrical frame*

The cylindrical coordinates of a point are $r_P$, $\theta$, $z$. The position of a point is defined by three scalar functions: polar radius $r_P(t)$, polar angle $\theta(t)$ and "height" $z(t)$ which are continuous and derivable at least to the second order.

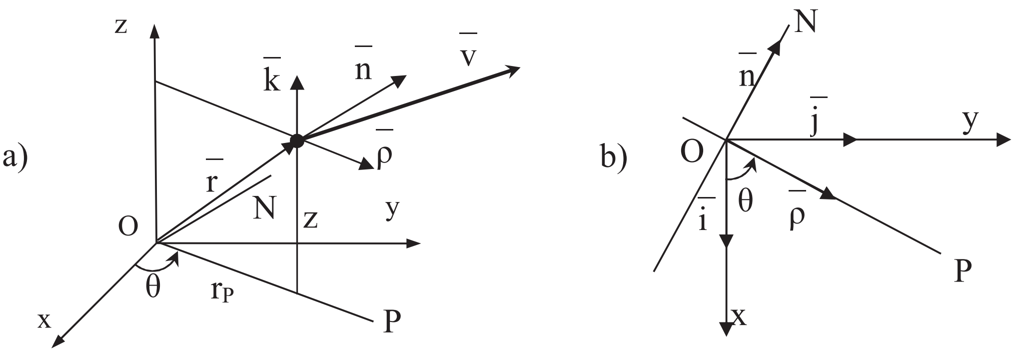



**Fig. 10.4 Cylindrical coordinates frame**

The axes OP, ON, Oz have the unit vectors $\bar{\rho}$, $\bar{n}$, $\bar{k}$. Since Oz is a fixed axis $\dot{\bar{k}} = 0$. The unit vectors $\bar{\rho}$ and $\bar{n}$ may be written:

$$\bar{\rho} = \cos\theta\bar{i} + \sin\theta\bar{j};$$
$$\bar{n} = -\sin\theta\bar{i} + \cos\theta\bar{j}; \quad (10.11)$$

It follows that:

$$\dot{\bar{\rho}} = \text{-}\sin\theta\dot{\theta}\bar{i} + \cos\theta\dot{\theta}\bar{j} = \dot{\theta}(\text{-}\sin\theta\bar{i} + \cos\theta\bar{j}) = \dot{\theta}\bar{n}$$
$$\dot{\bar{n}} = \text{-}\cos\theta\dot{\theta}\bar{i} \text{ - } \sin\theta\dot{\theta}\bar{j} = \text{-}\dot{\theta}(\cos\theta\bar{i} + \sin\theta\bar{j}) = \text{-}\dot{\theta}\bar{\rho} \quad (10.12)$$

The position vector $\bar{r}(t)$ has the expression:

$$\bar{r}(t) = r_P(t)\bar{\rho} + z(t)\bar{k}\,. \quad (10.13)$$

Note that $r(t)$ does not represent in general the modulus of $\bar{r}(t)$, with the exception of the case $z(t) \equiv 0$, which corresponds to a planar motion and the cylindrical coordinate system reduces to a **polar coordinate system**. The expressions of the velocity $\overline{v}$ and of the acceleration $\overline{a}$ become:

$$\begin{aligned}\overline{v}(t) &= \dot{\overline{r}}(t) = \dot{r}_P\overline{\rho} + r_P\dot{\overline{\rho}} + \dot{z}\overline{k} = \dot{r}_P\overline{\rho} + r_P\dot{\theta}\overline{n} + \dot{z}\overline{k} \\ \overline{a}(t) &= \dot{\overline{v}}(t) = \ddot{r}_P\overline{\rho} + \dot{r}_P\dot{\overline{\rho}} + \overline{n}(\dot{r}_P\dot{\theta} + r_P\ddot{\theta}) + r_P\dot{\theta}\dot{\overline{n}} + \ddot{z}\overline{k} \\ &= (\ddot{r}_P - r_P\dot{\theta}^2)\overline{\rho} + (r_P\ddot{\theta} + 2\dot{r}_P\dot{\theta})\overline{n} + \ddot{z}\overline{k}\end{aligned} \tag{10.14}$$

It follows that projection of the velocity and of the acceleration of the axes of a cylindrical system of coordinates are, after dropping the index *p* in order to comply with the notations of other authors:

$$\begin{cases} v_\rho = \dot{r} \\ v_n = r\dot{\theta} \\ v_z = \dot{z} \end{cases} \qquad \begin{cases} a_\rho = \ddot{r} - r\dot{\theta}^2 \\ a_n = r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ a_z = \ddot{z} \end{cases} \tag{10.15}$$

in which *r(t)* is the polar radius function.

## *10.7. Projections of velocity and acceleration on a Serret-Frenet frame*

A moving point describes a trajectory (Γ) as it moves. A point $M_0$ is selected (fixed) as origin on this trajectory (Fig. 10.5). The length of the arc $M_0M$ is denoted by *s*. The position vector can be expressed as the **intrinsic equation** of the path:

$$\overline{r} = \overline{r}(s), \tag{10.16}$$

and the scalar function

$$s = s(t), \tag{10.17}$$

will define the position of the point M on the trajectory.

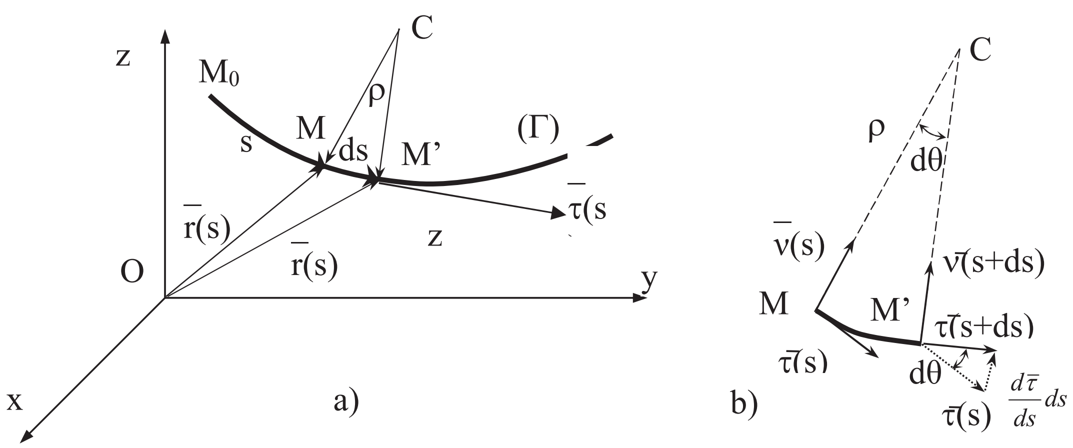



**Fig. 10.5 A moving point in the Serret – Frenet coordinate frame**

The coordinate frame of Serret-Frenet (called also natural or intrinsic) is a moving frame. Its axes are: the tangent, the principal normal and the binormal to the trajectory, with the origin in the moving point M. Denote by $\overline{\tau}, \overline{\nu}$ and $\overline{\beta}$ the unit vectors of these axes (Fig. 10.5b). The following formula provides the orientation of the tangent unit vector:

$$\lim_{M \to M'} \frac{\overline{MM'}}{\left|\overline{MM'}\right|} = \frac{d\overline{r}}{ds} = \overline{\tau} . \tag{10.18}$$

Three infinite close points on (Γ) e.g. M, M' and another point M" between them, are in general non-collinear. It means that these points define a plane, (osculating plane) and moreover a circle of center C (curvature center) and radius ρ (curvature radius). The arc length can thus be written as

$$ds = \rho d\theta . \tag{10.19}$$

As can be seen from (Fig. 10.5b), the two unit vectors $\overline{\tau}(s)$and $\overline{\tau}(s+ds)$make an infinitesimal angle $d\theta$ and in the isosceles triangle they form, the following formula holds:

$$d\theta \approx \sin d\theta = 2\frac{\frac{1}{2}\left|\frac{d\overline{\tau}}{ds}ds\right|}{\left|\overline{\tau}\right|} = \left|\frac{d\overline{\tau}}{ds}\right|ds \tag{10.20}$$

The consequence is that the modulus of the derivative of $\overline{\tau}(s)$ with respect to *s* is:

$$\left|\frac{d\overline{\tau}}{ds}\right| = \frac{1}{\rho} . \tag{10.21}$$

As for the orientation, in the triangle mentioned above (Fig. 10.5b) the vector $\frac{d\overline{\tau}}{ds}$ is perpendicular on the bisector of the angle θ. Since this angle is infinitesimal, the following formula can be deduced:

$$\frac{d\overline{\tau}}{ds} = \frac{1}{\rho}\overline{\nu} . \tag{10.22}$$

The third unit vector (of the binormal) is simply defined as $\overline{\beta} = \overline{\tau} \times \overline{\nu}$ . The velocity in this frame is

$$\overline{v} = \frac{d\overline{r}}{dt} = \frac{d\overline{r}}{ds}\frac{ds}{dt} = \dot{s}\overline{\tau} . \tag{10.23}$$

The acceleration of the point has the expression:

$$\overline{a} = \frac{d\overline{v}}{dt} = \ddot{s}\overline{\tau} + \dot{s}\frac{d\overline{\tau}}{dt} = \ddot{s}\overline{\tau} + \dot{s}\frac{d\overline{\tau}}{ds}\frac{ds}{dt} = \ddot{s}\overline{\tau} + \frac{\dot{s}^2}{\rho}\overline{\nu} . \tag{10.24}$$

It follows that the projections of the velocity and of the acceleration of a moving point on the axes of this frame are:

$$\begin{cases} v_\tau = \dot{s} \\ v_\nu = 0 \\ v_\beta = 0 \end{cases} \qquad \begin{cases} a_\tau = \ddot{s} \\ a_\nu = \dfrac{\dot{s}^2}{\rho} \\ a_\beta = 0 \end{cases} \tag{10.25}$$

### *10.8. Velocity and acceleration in a generalized orthogonal frame*

A general system of generalized orthogonal coordinates ($q_1, q_2, q_3$) is considered in the following. The position vector is then:

$$\overline{r} = \overline{r}\left(q_1, q_2, q_3\right). \tag{10.26}$$

During the motion, the three coordinates $q_1(t), q_2(t), q_3(t)$ are assumed continuous and derivable. The expression of the velocity $\overline{v}$ is:

$$\overline{v} = \frac{\partial \overline{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \overline{r}}{\partial q_2}\dot{q}_2 + \frac{\partial \overline{r}}{\partial q_3}\dot{q}_3. \tag{10.27}$$

The unit vectors of axes are:

$$\overline{e}_1 = \frac{\dfrac{\partial \overline{r}}{\partial q_1}}{\left|\dfrac{\partial \overline{r}}{\partial q_1}\right|}; \quad \overline{e}_2 = \frac{\dfrac{\partial \overline{r}}{\partial q_2}}{\left|\dfrac{\partial \overline{r}}{\partial q_2}\right|}; \quad \overline{e}_3 = \frac{\dfrac{\partial \overline{r}}{\partial q_3}}{\left|\dfrac{\partial \overline{r}}{\partial q_3}\right|}. \tag{10.28}$$

Denote by:

$$H_1 = \left|\frac{\partial \overline{r}}{\partial q_1}\right|; \quad H_2 = \left|\frac{\partial \overline{r}}{\partial q_2}\right|; \quad H_3 = \left|\frac{\partial \overline{r}}{\partial q_3}\right|, \tag{10.29}$$

which are named the **coefficients of Lamé**. The expression (10.27) may be written:

$$\overline{v} = H_1\dot{q}_1\overline{e}_1 + H_2\dot{q}_2\overline{e}_2 + H_3\dot{q}_3\overline{e}_3. \tag{10.30}$$

It follows that the projections of the velocity on the axes defined by the unit vectors ($\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$) are:

$$\begin{cases} v_1 = H_1\dot{q}_1 \\ v_2 = H_2\dot{q}_2 \\ v_3 = H_3\dot{q}_3 \end{cases} \tag{10.31}$$

The expression of acceleration is:

$$\overline{a} = \frac{d\overline{v}}{dt} \tag{10.32}$$

The projections of the acceleration on the axes defined above can be obtained using a property of the scalar product:

$$a_1 = \bar{a} \cdot \bar{e}_1; \quad a_2 = \bar{a} \cdot \bar{e}_2; \quad a_3 = \bar{a} \cdot \bar{e}_3 . \tag{10.33}$$

The expression of $a_1$ for example, may be written successively:

$$a_1 = \bar{a} \cdot \bar{e}_1 = \frac{d\bar{v}}{dt} \frac{1}{H_1} \frac{\partial \bar{r}}{\partial q_1} = \frac{1}{H_1} \left[ \frac{d}{dt} \left( \bar{v} \frac{\partial \bar{r}}{\partial q_1} \right) - \bar{v} \frac{d}{dt} \left( \frac{\partial \bar{r}}{\partial q_1} \right) \right] \tag{10.34}$$

From (10.27) it can be deduced, considering $q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3$ as independent variables:

$$\frac{\partial \bar{r}}{\partial q_1} = \frac{\partial \bar{v}}{\partial \dot{q}_1} \tag{10.35}$$

The following relation is also necessary:

$$\frac{d}{dt} \left( \frac{\partial \bar{r}}{\partial q_1} \right) = \frac{\partial}{\partial q_1} \left( \frac{d\bar{r}}{dt} \right) = \frac{\partial \bar{v}}{\partial q_1} \tag{10.36}$$

Hence

$$a_1 = \frac{1}{H_1} \left[ \frac{d}{dt} \left( \bar{v} \frac{\partial \bar{v}}{\partial q_1} \right) - \bar{v} \frac{\partial \bar{v}}{\partial q_1} \right] = \frac{1}{2H_1} \left[ \frac{d}{dt} \left( \frac{\partial (v^2)}{\partial \dot{q}_i} \right) - \frac{\partial (v^2)}{\partial q_1} \right] \tag{10.37}$$

Analogous expressions can be obtained for $a_2$ and $a_3$. The projections of the velocity and of the acceleration on the axes defined above, can be cast in index form:

$$\begin{aligned} v_i &= H_i \dot{q}_i ; \\ a_i &= \frac{1}{2H_i} \left[ \frac{d}{dt} \left( \frac{\partial v^2}{\partial \dot{q}_i} \right) - \frac{\partial v^2}{\partial q_i} \right]; \quad i = 1,2,3 \end{aligned} \tag{10.38}$$

in which, using the Cartesian projections of the position vector, $\bar{r} = x(q_1, q_2, q_3)\bar{i} + y(q_1, q_2, q_3)\bar{j} + z(q_1, q_2, q_3)\bar{k}$ , the coefficients of Lamé are:

$$H_i = \left| \frac{\partial \bar{r}}{\partial q_i} \right| = \sqrt{\left( \frac{\partial x}{\partial q_i} \right)^2 + \left( \frac{\partial y}{\partial q_i} \right)^2 + \left( \frac{\partial z}{\partial q_i} \right)^2}; \quad i = 1,2,3 \tag{10.39}$$

Example. Spherical coordinates

Write the velocity and acceleration of a particle in spherical coordinates $r, \theta, \varphi$ which are independent functions of time (Fig. 10.6).

The relations between the Cartesian coordinates *x, y, z* and $r, \theta, \varphi$ are:

$$x = r \sin\theta \cos\varphi; \quad y = r \sin\theta \sin\varphi; \quad z = r \cos\theta . \tag{10.40}$$

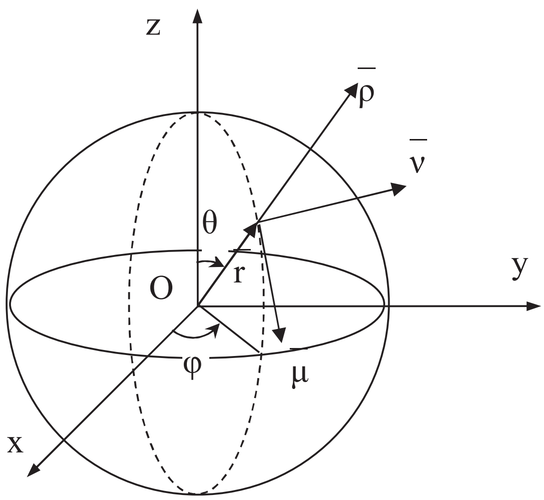



**Fig. 10.6 Motion in spherical coordinates**

Using (10.39), the coefficients of Lamé are:

$$H_1 = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1; \quad H_2 = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r;$$
$$H_3 = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = r\sin\theta. \tag{10.41}$$

The components of the velocity from (10.31) can be deduced as:

$$\begin{cases} v_\rho \equiv v_r = H_1\dot{r} = \dot{r} \\ v_\mu \equiv v_\theta = H_2\dot{\theta} = r\dot{\theta} \\ v_\nu \equiv v_\varphi = H_3\dot{\varphi} = r\sin\theta\,\dot{\varphi} \end{cases} \Rightarrow \quad v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta \tag{10.42}$$

The derivatives required by (10.38) are:

$$\frac{\partial v^2}{\partial \dot{r}} = 2\dot{r}; \qquad \frac{\partial v^2}{\partial \dot{\theta}} = 2r^2\dot{\theta}; \qquad \frac{\partial v^2}{\partial \dot{\varphi}} = 2r^2\dot{\varphi}\sin^2\theta;$$
$$\frac{\partial v^2}{\partial r} = 2\left(r\dot{\theta}^2 + r\dot{\varphi}^2\sin^2\theta\right); \quad \frac{\partial v^2}{\partial \theta} = 2r^2\dot{\varphi}^2\sin\theta\cos\theta; \quad \frac{\partial v^2}{\partial \varphi} = 0. \tag{10.43}$$

The acceleration components from (10.38) can then be deduced in this case:

$$a_\rho \equiv a_r = \frac{1}{2}\left[\frac{d}{dt}\left(\frac{\partial v^2}{\partial \dot{r}}\right) - \frac{\partial v^2}{\partial r}\right] = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta;$$
$$a_\mu \equiv a_\theta = \frac{1}{2r}\left[\frac{d}{dt}\left(\frac{\partial v^2}{\partial \dot{\theta}}\right) - \frac{\partial v^2}{\partial \theta}\right] = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta; \tag{10.44}$$
$$a_\nu \equiv a_\varphi = \frac{1}{2r\sin\theta}\left[\frac{d}{dt}\left(\frac{\partial v^2}{\partial \dot{\varphi}}\right) - \frac{\partial v^2}{\partial \varphi}\right] = 2\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta.$$

### *10.9. Angular velocity and acceleration. Areal velocity*

A point moves on a trajectory (Γ). Two close positions of the point on (Γ) are marked by *M* and respectively *N*. The angle *MON* is denoted by Δθ, and the area of the triangle *MON* is denoted by ΔA.

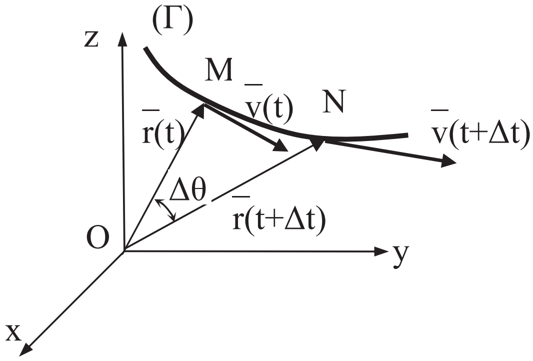



**Fig. 10.7 Angular velocity and acceleration.**

By definition the modulus of the **angular velocity** and **angular acceleration** are:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}; \quad \varepsilon = \frac{d\omega}{dt}. \tag{10.45}$$

The **areal velocity** is by definition:

$$\Omega = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t}. \tag{10.46}$$

Admitting that $OM \le ON$, it can be written the inequality:

$$\frac{1}{2} OM \cdot OM \cdot \Delta\theta \le \Delta A \le \frac{1}{2} ON \cdot ON \cdot \Delta\theta. \tag{10.47}$$

Dividing this inequality by Δt and passing to the limit for Δt →0, the inequality becomes:

$$\lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} \le \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} \le \lim_{\Delta t \to 0} \frac{1}{2} \left(r + \Delta r\right)^2 \frac{\Delta\theta}{\Delta t}. \tag{10.48}$$

But $\lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} \left(r + \Delta r\right)^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \dot{\theta}^2$, so that

$$\Omega = \frac{1}{2} r^2 \dot{\theta}^2 \tag{10.49}$$

which represents another definition of the areal velocity. The areal velocity is used in expressing Kepler laws in chapter 13.

## *10.10. Particular motions of a point*

In this paragraph are defined the most important particular motions and are deduced simpler kinematic formulas for the velocity and acceleration of a point in these cases.

### 10.10.1. Uniform rectilinear motion

By definition the trajectory of this motion is a straight line (Fig. 10.8) and the scalar of the velocity is constant. Taking the *Ox* axis along this trajectory, it can be written:

$$\dot{x} = v; \quad y \equiv 0; \quad z \equiv 0. \tag{10.50}$$

Integrating this differential equation, the projections of the position vector can be determined:

$$x = vt + C_1. \tag{10.51}$$

If at the initial moment (*t=0*) $x = x_0$, then the motion is along the *Ox* axis, with the position, velocity and acceleration:

$$x = vt + x_0; \quad \dot{x} = v; \quad \ddot{x} = 0. \tag{10.52}$$

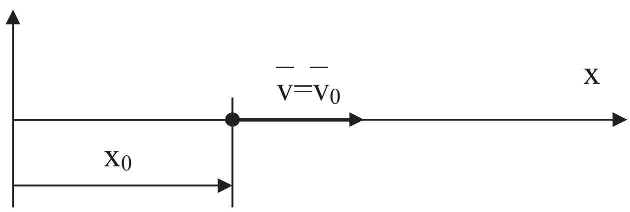



**Fig. 10.8 Rectilinear motion of a point. Initial conditions.**

### 10.10.2. Uniformly accelerated rectilinear motion

By definition the trajectory of this line is a straight line (Fig. 10.8) and the scalar of the acceleration is constant. If *Ox* is the trajectory then:

$$\ddot{x} = a; \quad y \equiv 0; \quad z \equiv 0. \tag{10.53}$$

By successive integration the velocity and the position of the point are:

$$\dot{x} = at + C_1; \quad x = a\frac{t^2}{2} + C_1 t + C_2 \tag{10.54}$$

If at the initial moment (*t=0*) $x = x_0$, $v = v_0$ then the motion is along the *Ox* axis, with position, velocity and acceleration given by formulas:

$$x = \frac{1}{2}at^2 + v_0 t + x_0; \quad \dot{x} = v = at + v_0; \quad \ddot{x} = a \tag{10.55}$$

### 10.10.3. Circular motion

A point M moves along a circle of radius $R$ and of center $O$. A positive sense on the circle is selected and an initial point $M_0$ is given.

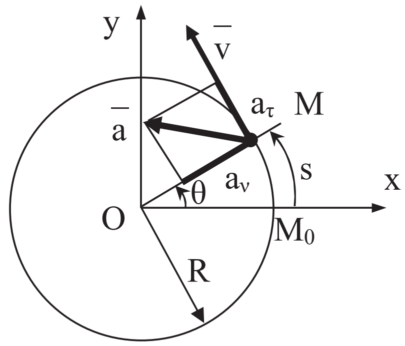



**Fig. 10.9 Circular motion of the point**

The angular velocity is $\omega = \dot{\theta}$ and the angular acceleration is $\varepsilon = \dot{\omega} = \ddot{\theta}$. The velocity and the acceleration can be expressed using the Serret-Frenet frame:

$$
\begin{aligned}
v &= \dot{s} = R\dot{\theta} = R\omega \\
a_\tau &= \ddot{s} = R\ddot{\theta} = R\varepsilon \\
a_\nu &= \frac{\dot{s}^2}{\rho} = \frac{(R\omega)^2}{R} = R\omega^2
\end{aligned}
\qquad (10.56)
$$

The orientations of the corresponding components, assumed to be positive, are shown in Fig. 10.9.

Note: In most technical applications the angular velocity is indicated in rotations per minute (RPM), which corresponds to a value in International System (SI) of

$$\omega = \frac{\pi \cdot (RPM)}{30}. \qquad (10.57)$$

### 10.10.4. Motion along a cycloid

A circle of radius $R$ is rolling along a straight line. The motion of an arbitrary point on the circumference of the circle is followed (Fig. 10.10). It is assumed that at the initial moment the point $M$ is a point of tangency of the circle and the straight line $Ox$ at the origin of a coordinate system. If the circle turns through an angle $\theta$, the rolling distance $OP = PM = R\theta$ and the expressions for the coordinates $x$ and $y$ of M can be written as:

$$
\begin{aligned}
x &= OP - MQ = R\theta - R\sin\theta \\
y &= PQ = CP - CQ = R - R\cos\theta
\end{aligned}
\qquad (10.58)
$$

The path of the point M is named a cycloid. Supposing that the circle revolves uniformly i.e. that the angle $\theta = \omega t$ (where $\omega$ is a constant), the equations of the motion of the point M are:

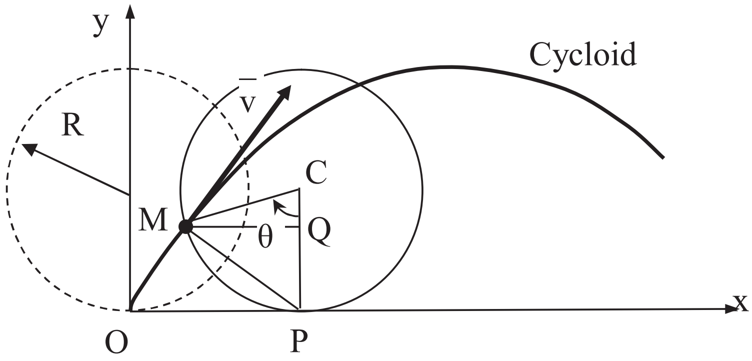



**Fig. 10.10. Motion along a cycloid**

$$\begin{aligned} x &= R(\omega t - \sin\omega t) \\ y &= R(1-\cos\omega t) \end{aligned} \qquad (10.59)$$

The derivatives with respect to time provide the components of velocity:

$$\begin{aligned} \dot{x} &= R\omega(1-\cos\omega t) \\ \dot{y} &= R\omega\sin\omega t \end{aligned}, \qquad (10.60)$$

and the derivatives of the velocities represent the components of acceleration:

$$\begin{aligned} \ddot{x} &= R\omega^2 \sin\omega t \\ \ddot{y} &= R\omega^2 \cos\omega t \end{aligned}. \qquad (10.61)$$

Hence

$$\begin{aligned} |\overline{v}| &= \sqrt{R^2\omega^2(1-\cos\omega t)^2 + R^2\omega^2\sin^2\omega t} = 2R\omega\left|\sin\frac{\omega t}{2}\right| = PM\,\omega \\ |\overline{a}| &= \sqrt{R^2\omega^4\left(\sin^2\omega t + \cos^2\omega t\right)} = R\omega^2 = CM\,\omega^2 \end{aligned} . \qquad (10.62)$$

It can be proven that the velocity can be obtained as for an instantaneous rotation around P, and the acceleration is obtained as for an instantaneous rotation around C in the hypothesis $\theta = \omega t$. The moduli of the two vectors (10.62) have this property. Remains to prove $\overline{v} \perp \overline{PM}$ and $\overline{a} \parallel \overline{CM}$.

The vectors $\overline{PM} = (x_M - x_P)\overline{i} + (y_M - y_P)\overline{j} = -R\sin\omega t\overline{i} + (R - R\cos\omega t)\overline{j}$ and

$\overline{CM} = (x_M - x_C)\overline{i} + (y_M - y_C)\overline{j} = -R\sin\omega t\overline{i} + (-R\cos\omega t)\overline{j}$, can be used for the scalar product to prove the perpendicularity:

$$\overline{v} \cdot \overline{PM} = -R\omega(1-\cos\omega t)R\sin\omega t + R\omega\sin\omega t \cdot R(1-\cos\omega t) = 0. \quad (10.63)$$

and the parallelism can be proven by proportionality of the components:

$$\frac{R\omega^2\sin\omega t}{-R\sin\omega t} = \frac{R\omega^2\cos\omega t}{-R\cos\omega t}. \quad (10.64)$$

It can also be proven that at the points of return (y=0), the velocity is null and the acceleration is along the Oy direction. Indeed:

$$y = 0 \;\Rightarrow R(1-\cos\omega t) = 0 \;\Rightarrow \omega t = 2k\pi;\;\; k = 1,2,... \quad (10.65)$$

The velocity has the components:

$$\begin{aligned} \dot{x} &= R\omega(1-\cos\omega t) = 0 \\ \dot{y} &= R\omega\sin\omega t = R\omega\sin(2k\pi) = 0 \end{aligned} \quad (10.66)$$

The acceleration is defined by:

$$\begin{aligned} \ddot{x} &= R\omega^2\sin(2k\pi) = 0 \\ \ddot{y} &= R\omega^2\cos(2k\pi) = R\omega^2 \end{aligned} \quad (10.67)$$

This result indicates that the path is perpendicular on the Ox axis in these points.

### 10.10.5. Motion along a helix

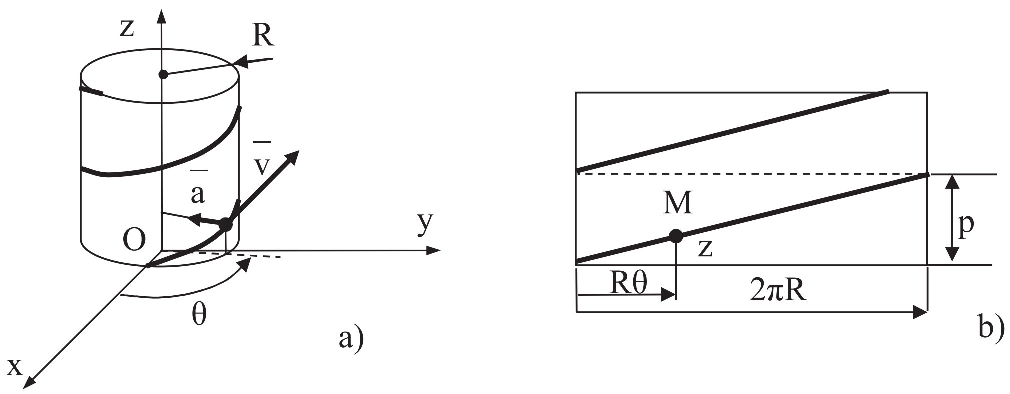



**Fig. 10.11 Motion along a helix**

A helix on a circular cylinder of radius R is considered. By definition the helix crosses an arbitrary generatrix of the cylinder in equidistant points (Fig. 10.11a).
Let *p* be the distance between two such successive points, called **lead** or **pitch**. Cutting the cylindrical surface along a generatrix and unfolding this surface will generate a rectangle. The helix appears as parallel straight lines (Fig. 10.11b). The coordinate z may be determined from the obvious proportion:

$$\frac{z}{R\theta}=\frac{p}{2\pi R} \quad \Rightarrow \; z=\frac{p}{2\pi}\theta \tag{10.68}$$

It follows that the parametric equations of the helix are:

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \\ z &= \frac{p}{2\pi}\theta \end{aligned} \tag{10.69}$$

Assuming that the angle θ is proportional to the time t, i.e. $\theta=\omega t$ with ω a constant, the equations of motion become:

$$\begin{aligned} x &= R\cos\omega t \\ y &= R\sin\omega t \\ z &= \frac{p}{2\pi}\omega t \end{aligned} \tag{10.70}$$

The components of the velocity and acceleration are obtained as successive derivatives of these expressions:

$$\overline{v}\begin{cases} \dot{x}=-R\omega\sin\omega t \\ \dot{y}=R\omega\cos\omega t \\ \dot{z}=\dfrac{p}{2\pi}\omega \end{cases} ; \qquad |\overline{v}|=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}=\omega\sqrt{R^2+\frac{p^2}{4\pi^2}} \tag{10.71}$$

$$\overline{a}\begin{cases} \ddot{x}=-R\omega^2\cos\omega t \\ \ddot{y}=R\omega^2\sin\omega t \\ \ddot{z}=0 \end{cases} ; \qquad |\overline{a}|=\sqrt{\ddot{x}^2+\ddot{y}^2+\ddot{z}^2}=R\omega^2 \tag{10.72}$$

Hence, in the assumed hypothesis, the moduli of the velocity and acceleration are constant.

Note. Since $|\overline{v}|$ is constant, the acceleration can only be perpendicular to the helix and its expression is $\frac{v^2}{\rho}=|\overline{a}|$ according to (10.25). It can be thus obtained the curvature radius of the helix:

$$\rho=\frac{v^2}{|\overline{a}|}=\frac{\omega^2\left(R^2+\dfrac{p^2}{4\pi^2}\right)}{R\omega^2}=R+\frac{p^2}{4\pi^2 R} \tag{10.73}$$

# 11. KINEMATICS OF A RIGID BODY

## *11.1. Preliminaries*

A rigid body is moving with respect to a fixed Cartesian frame $O_1x_1y_1z_1$. The motion of the rigid body is completely defined if it is possible to determine the motions of each point of the rigid body with respect to the chosen Cartesian frame. Let *M* be an arbitrary point of the rigid body. In order to define the position of this point with respect to the rigid body, it is necessary to consider a second Cartesian frame $Oxyz$ attached to the rigid body. Denote $\overline{OM} = \overline{r}$, $\overline{O_1M} = \overline{r}_1$ and $\overline{O_1O} = \overline{r}_O$ (Fig. 11.1).

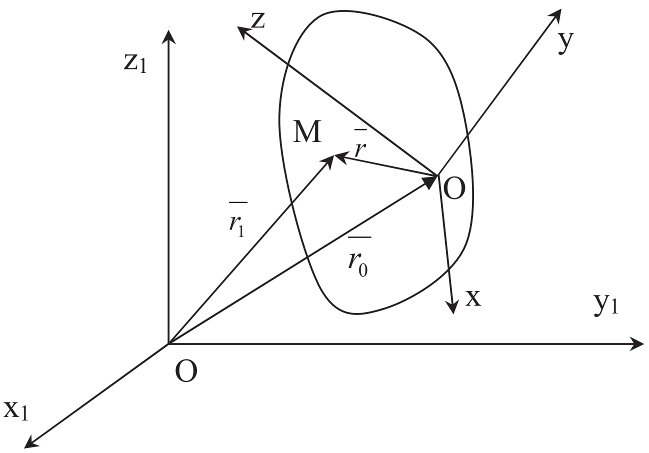



**Fig. 11.1 Motion of a rigid body**

The following relation is obvious:

$$\overline{r}_1(t) = \overline{r}_0(t) + \overline{r}(t);\quad \forall t. \tag{11.1}$$

Since $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$, the previous expression becomes:

$$\overline{r}_1 = \overline{r}_0 + x\overline{i} + y\overline{j} + z\overline{k} \tag{11.2}$$

Note that Cartesian frame $Oxyz$ is evidently, a movable one, but the coordinates *x, y, z* are constant, because the Cartesian frame $Oxyz$ is attached to the rigid body.

## *11.2. The field of velocities. Euler's formula*

The derivative of expression (11.2) with respect to time represents the expression of the velocity for a certain point of the rigid body in the form:

$$\overline{v} = \overline{v}_0 + x\dot{\overline{i}} + y\dot{\overline{j}} + z\dot{\overline{k}}, \tag{11.3}$$

because $\overline{v} = \dot{\overline{r}}_1$ is the velocity of a certain point of the rigid body with respect to the fixed Cartesian frame $O_1x_1y_1z_1$ and $\overline{v}_0 = \dot{\overline{r}}_0$ is the velocity of the origin of the moving Cartesian frame *Oxyz* with respect to the frame $O_1x_1y_1z_1$.

### 11.2.1. Formulas of Poisson

In order, to express the projections of the unit vectors $\dot{\bar{i}}, \dot{\bar{j}}, \dot{\bar{k}}$ on the axes Ox, Oy and Oz of movable Cartesian frame Oxyz, the following nine scalar products are determined:

| | Ox | Oy | Oz |
|---|---|---|---|
| $\dot{\bar{i}}$ | $\dot{\bar{i}}\bar{i}$ | $\dot{\bar{i}}\bar{j}$ | $\dot{\bar{i}}\bar{k}$ |
| $\dot{\bar{j}}$ | $\dot{\bar{j}}\bar{i}$ | $\dot{\bar{j}}\bar{j}$ | $\dot{\bar{j}}\bar{k}$ |
| $\dot{\bar{k}}$ | $\dot{\bar{k}}\bar{i}$ | $\dot{\bar{k}}\bar{j}$ | $\dot{\bar{k}}\bar{k}$ |

The vector functions $\bar{i} = \bar{i}(t), \bar{j} = \bar{j}(t)$ and $\bar{k} = \bar{k}(t)$ satisfy these six conditions

$$\begin{aligned} &\bar{i}^2 = 1; \quad \bar{j}^2 = 1; \quad \bar{k}^2 = 1 \\ &\bar{i} \cdot \bar{j} = 0; \ \bar{j} \cdot \bar{k} = 0; \ \bar{k} \cdot \bar{i} = 0 \end{aligned} \tag{11.4}$$

Differentiating these relations, it follows

$$\begin{aligned} &2\bar{i} \cdot \dot{\bar{i}} = 0; \ 2\bar{j} \cdot \dot{\bar{j}} = 0; \ 2\bar{k} \cdot \dot{\bar{k}} = 0; \\ &\dot{\bar{i}} \cdot \bar{j} + \bar{i} \cdot \dot{\bar{j}} = 0; \ \dot{\bar{j}} \cdot \bar{k} + \bar{j} \cdot \dot{\bar{k}} = 0; \ \dot{\bar{k}} \cdot \bar{i} + \bar{k} \cdot \dot{\bar{i}} = 0. \end{aligned} \tag{11.5}$$

It follows that

$$\begin{aligned} &\bar{i} \cdot \dot{\bar{i}} = 0; \ \bar{j} \cdot \dot{\bar{j}} = 0; \ \bar{k} \cdot \dot{\bar{k}} = 0; \\ &\dot{\bar{i}} \cdot \bar{j} = -\bar{i} \cdot \dot{\bar{j}} = \omega_z; \ \dot{\bar{j}} \cdot \bar{k} = -\bar{j} \cdot \dot{\bar{k}} = \omega_x; \ \dot{\bar{k}} \cdot \bar{i} = -\bar{k} \cdot \dot{\bar{i}} = \omega_y. \end{aligned} \tag{11.6}$$

In which $\omega_x$, $\omega_y$, $\omega_z$ are three scalar functions. It follows that the above table is a skew symmetric one and the expressions of $\dot{\bar{i}}, \dot{\bar{j}}, \dot{\bar{k}}$ may be written:

$$\dot{\bar{i}} = \omega_z \bar{j} - \omega_y \bar{k}; \ \dot{\bar{j}} = \omega_x \bar{k} - \omega_z \bar{i}; \ \dot{\bar{k}} = \omega_y \bar{i} - \omega_z \bar{j} . \tag{11.7}$$

It can be proven that these expressions may also be written in the form

$$\dot{\bar{i}} = \bar{\omega} \times \bar{i}; \quad \dot{\bar{j}} = \bar{\omega} \times \bar{j}; \quad \dot{\bar{k}} = \bar{\omega} \times \bar{k}, \tag{11.8}$$

in which $\bar{\omega}$ is the vector defined by:

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k} . \tag{11.9}$$

The results (11.8) represent the formulas of Poisson for the derivatives of unit vectors. The expression (11.3) of the velocity may be written:

$$\bar{v} = \bar{v}_0 + x\bar{\omega} \times \bar{i} + y\bar{\omega} \times \bar{j} + z\bar{\omega} \times \bar{k} = \bar{v}_0 + \bar{\omega} \times (x\bar{i} + y\bar{j} + z\bar{k}) \tag{11.10}$$

or simpler:

$$\overline{v} = \overline{v}_0 + \overline{\omega} \times \overline{r} \tag{11.11}$$

This is **Euler's Formula** for the **field of velocities** in the most general motion of a rigid body. Comparing formulas (11.3) and (11.10) it follows that $\dot{\overline{r}} = \overline{\omega} \times \overline{r}$.

## *11.3. The field of accelerations. Rivals Formula*

Derivation of the formula (11.10) produces the acceleration

$$\overline{a} = \overline{a}_0 + \dot{\overline{\omega}} \times \overline{r} + \overline{\omega} \times \dot{\overline{r}}. \tag{11.12}$$

By definition $\dot{\overline{v}} = \overline{a}$ is the acceleration of a certain point of the rigid body with respect to the fixed Cartesian frame $O_1x_1y_1z_1$ and $\dot{\overline{v}}_0 = \overline{a}_0$ is the acceleration of the origin O of the movable Cartesian frame Oxyz with respect to the frame $O_1x_1y_1z_1$. It has been proven before that $\dot{\overline{r}} = \overline{\omega} \times \overline{r}$, and using the notation $\dot{\overline{\omega}} = \overline{\varepsilon}$, the formula (11.12) becomes:

$$\overline{a} = \overline{a}_0 + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times \left( \overline{\omega} \times \overline{r} \right). \tag{11.13}$$

This expression is known as **Rivals formula** for the **field of accelerations** in the most general motion of a rigid body.

## *11.4. Particular motions of a rigid body*

In this paragraph are deduced particular formulas for the velocity and acceleration of an arbitrary point belonging to a moving rigid body.

### 11.4.1. Translation

By definition a rigid body is in motion of translation if a geometric vector AB joining two arbitrary points A and B of the body maintains its direction and sense during the motion of the body.

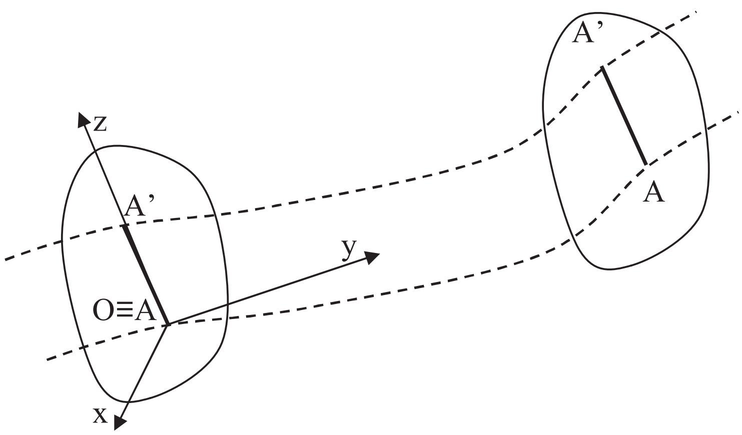



**Fig. 11.2 Translation motion**

Note that the path of a point of a body in translation may be an arbitrary one. The paths of all points are congruent and they can be superimposed by means of a translation (Fig. 11.2). In particular the unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ of the axes of the moving Cartesian frame are constant. It follows that $\dot{\bar{i}} = \bar{0},\ \dot{\bar{j}} = \bar{0},\ \dot{\bar{k}} = \bar{0}$ and consequently $\omega_x = \omega_y = \omega_z = 0$ and

$$\bar{\omega} = \bar{0} \Rightarrow \bar{\varepsilon} = \dot{\bar{\omega}} = \bar{0}. \tag{11.14}$$

The Euler and Rivals formulas become:

$$\bar{v} = \bar{v}_0(t); \quad \bar{a} = \bar{a}_0(t). \tag{11.15}$$

It follows that the fields of velocities and accelerations are fields of constant values at any given moment of time. This means that at a moment $t_1$ all the points of the rigid body have the same velocity and acceleration and this is true at any other moment $t_2$, but the effective value can change from one moment to another. The velocities and accelerations are in this case examples of free vectors.

### 11.4.2. Rotation about a fixed axis

By definition a rotation about an axis takes place if there are two fixed points of the rigid body during its motion. The two points define the axis of rotation and all the points of the axis keep their positions during motion. Being a rigid body, every point describes circles around the axis of rotation. It can be taken the *Oz* as rotation axis and the fixed Cartesian frame with $O_1z_1$ coincident with *Oz*. The angle between axes *Ox* and $O_1x_1$ is $\theta(t)$ and defines the rotation of the rigid body around the $O_1z_1$ axis (Fig. 11.3).The unit vectors of the mobile frame are $\bar{i}, \bar{j}, \bar{k}$ and those of the fixed frame $\bar{i}_1, \bar{j}_1, \bar{k}_1$.

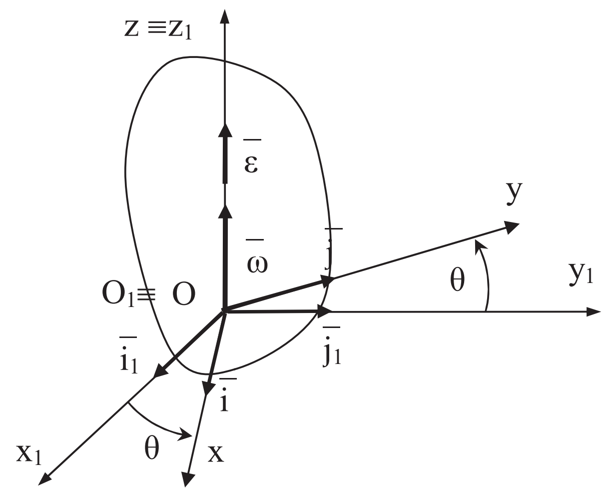



**Fig. 11.3 Rotation about a fixed axis**

The mobile unit vectors can be projected on the fixed axes:

$$\begin{aligned}\overline{i} &= \cos\theta\overline{i}_1 + \sin\theta\overline{j}_1 \\ \overline{j} &= -\sin\theta\overline{i}_1 + \cos\theta\overline{j}_1 \, . \\ \overline{k} &= \overline{k}_1\end{aligned} \tag{11.16}$$

The time derivatives of these vectors are:

$$\begin{aligned}\dot{\overline{i}} &= (-\sin\theta)\dot{\theta}\overline{i}_1 + (\cos\theta)\dot{\theta}\overline{j}_1 = \dot{\theta}\overline{j} \\ \dot{\overline{j}} &= (-\cos\theta)\dot{\theta}\overline{i}_1 + (-\sin\theta)\dot{\theta}\overline{j}_1 = -\dot{\theta}\overline{i} \\ \dot{\overline{k}} &= \overline{0}\end{aligned} \tag{11.17}$$

The components of the $\overline{\omega}$ vector are

$$\begin{aligned}\omega_x &= \dot{\overline{j}} \cdot \overline{k} = 0 \\ \omega_y &= \dot{\overline{k}} \cdot \overline{i} = 0 \, . \\ \omega_z &= \dot{\overline{i}} \cdot \overline{j} = \dot{\theta}\end{aligned} \tag{11.18}$$

Consequently

$$\overline{\omega} = \dot{\theta}\overline{k} = \omega\overline{k} \, . \tag{11.19}$$

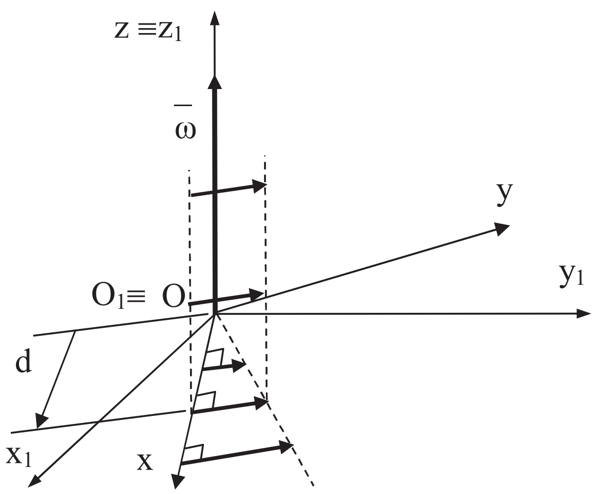



**Fig. 11.4 Velocity field for a rotation about a fixed axis**

Euler formula becomes in this case ($\overline{v}_0 = \overline{0}$) :

$$\overline{v} = \overline{\omega} \times \overline{r} = \begin{vmatrix}\overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & \omega \\ x & y & z\end{vmatrix} = -\omega y\overline{i} + \omega x\overline{j} \, , \tag{11.20}$$

from which the components of the velocity on the moving frame Oxyz are:

$$\bar{v}\begin{cases} v_x = -\omega y \\ v_y = \omega x \\ v_z = 0 \end{cases}. \tag{11.21}$$

It follows that the velocities are the same along a line parallel to the rotation axis, since the components do not depend on the $z$ coordinate (Fig. 11.4). Another property is the linear dependence on the distance $d$ to the rotation axis:

$$|\bar{v}| = \sqrt{\omega^2 x^2 + \omega^2 y^2} = \omega d\,. \tag{11.22}$$

The Rivals formula, takes into account the obvious facts: $\bar{a}_0 = \bar{0}$; $\bar{\omega} = \omega\bar{k}$; $\bar{\varepsilon} = \varepsilon\bar{k}$, so that the acceleration field is given by:

$$\bar{a} = \varepsilon\bar{k}\times\bar{r} + \omega\bar{k}\times\left(\omega\bar{k}\times\bar{r}\right) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \varepsilon \\ x & y & z \end{vmatrix} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} \tag{11.23}$$

$$= \varepsilon\left(-y\bar{i} + x\bar{j}\right) - \omega^2\left(x\bar{i} + y\bar{j}\right) = \left(-\varepsilon y - \omega^2 x\right)\bar{i} + \left(\varepsilon x - \omega^2 y\right)\bar{j}$$

The projections of the acceleration on the axes of the moving frame Oxyz are:

$$\begin{aligned} a_x &= -\varepsilon y - \omega^2 x \\ a_y &= \varepsilon x - \omega^2 y \\ a_z &= 0 \end{aligned}\ . \tag{11.24}$$

It follows that the acceleration has the same value at a given instant for all the points along a line parallel with the rotation axis, since there is no variable $z$ in these formulas. The modulus of the acceleration increases linearly with increasing distance to the rotation axis (Fig. 11.5).

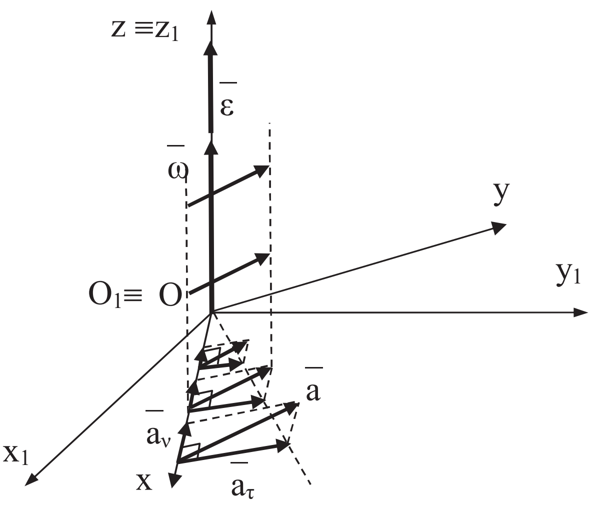



**Fig. 11.5 Acceleration field for the rotation about an axis**

As can be seen from (11.23) the acceleration can be expressed by two components:
- a tangent component

$$\overline{a}_\tau = \varepsilon\left(-y\overline{i} + x\overline{j}\right) \Rightarrow \left|\overline{a}_\tau\right| = \left|\varepsilon\right|\sqrt{x^2 + y^2} = \left|\varepsilon\right| d\,, \tag{11.25}$$

- a normal component

$$\overline{a}_\nu = -\omega^2\left(x\overline{i} + y\overline{j}\right) \ \left|\overline{a}_\nu\right| = \omega^2\sqrt{x^2 + y^2} = \omega^2 d\,, \tag{11.26}$$

in which *d* is again the distance to the rotation axis.

#### 11.4.3. Helical motion

By definition a rigid body is in helical motion if there exist two points of the rigid body remaining on a given straight line. It is easy to prove that the motion consists of a translation along the given line superimposed on a rotation around the same line taken as axis. Being a rigid body, every point describes helixes around the axis of rotation. It can be taken the *Oz* as rotation axis and the fixed Cartesian frame with $O_1z_1$ coincident with *Oz*. The angle between axes *Ox* and $O_1x_1$ is θ(t) and defines the rotation of the rigid body around the $O_1z_1$ axis and the distance between $O_1$ and *O* is denoted by $z_0$ (Fig. 11.6). The unit vectors of the mobile frame are $\overline{i}, \overline{j}, \overline{k}$ and those of the fixed frame $\overline{i}_1, \overline{j}_1, \overline{k}_1$ and their relative angular orientation is the same as for rotation.

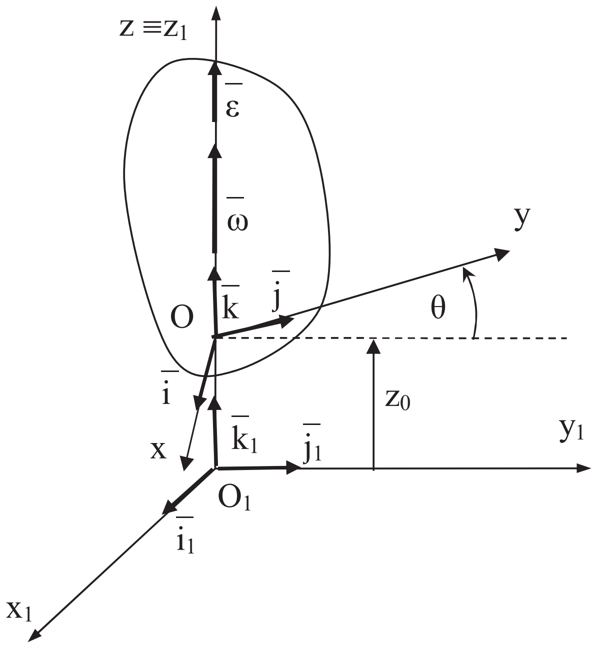



**Fig. 11.6 Helical motion**

Consequently the angular velocity and the axial velocity $\overline{v}_a = \overline{v}_O$ are

$$\overline{\omega} = \dot{\theta}\overline{k} = \omega\overline{k}; \quad \overline{v}_O = \dot{z}\overline{k} = v_0\overline{k} \overset{not.}{=} \overline{v}_a\,. \tag{11.27}$$

Euler formula becomes in this case:

$$\overline{v} = v_O\overline{k} + \overline{\omega}\times\overline{r} = v_O\overline{k} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y\,\overline{i} + \omega x\,\overline{j} + v_O\overline{k}\,, \tag{11.28}$$

from which the components of the velocity on the moving frame *Oxyz* are:

$$\overline{v}\begin{cases} v_x = -\omega y \\ v_y = \omega x \\ v_z = v_O \end{cases}. \tag{11.29}$$

It follows that the velocities are the same along a line parallel to the rotation axis, since the components do not depend on the *z* coordinate (Fig. 11.7). Another property is the linear dependence of the tangent component with the distance *d* to the rotation axis:

$$|\overline{v}_t| = \sqrt{v_x^2 + v_y^2} = \sqrt{\omega^2 x^2 + \omega^2 y^2} = \omega d\,. \tag{11.30}$$

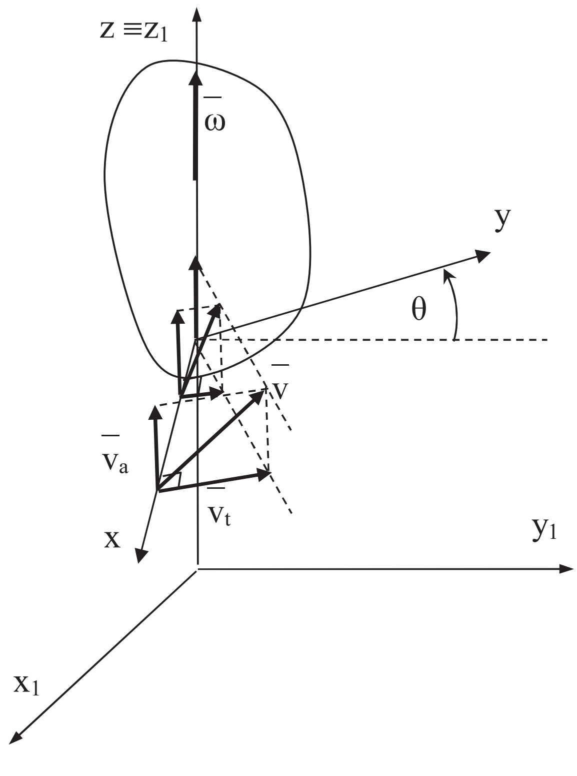



**Fig. 11.7 Velocity field for the helical motion**

The Rivals formula, takes into account the obvious facts: $\overline{a}_O = \ddot{z}\overline{k} = a_O\overline{k}$; $\overline{\omega} = \omega\overline{k}$; $\overline{\varepsilon} = \dot{\overline{\omega}} = \varepsilon\overline{k}$, so that the acceleration field is given by:

$$\overline{a} = a_O\overline{k} + \varepsilon\overline{k}\times\overline{r} + \omega\overline{k}\times\left(\omega\overline{k}\times\overline{r}\right) = a_O\overline{k} + \begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 0 & 0 & \varepsilon \\ x & y & z\end{vmatrix} + \begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 0 & 0 & \omega \\ -\omega y & \omega x & 0\end{vmatrix} \tag{11.31}$$

$$= \varepsilon\left(-y\overline{i} + x\overline{j}\right) - \omega^2\left(x\overline{i} + y\overline{j}\right) + a_O\overline{k} = \left(-\varepsilon y - \omega^2 x\right)\overline{i} + \left(\varepsilon x - \omega^2 y\right)\overline{j} + a_O\overline{k}$$

The projections of the acceleration on the axes of the moving frame Oxyz are:

$$\begin{aligned} a_x &= -\varepsilon y - \omega^2 x \\ a_y &= \varepsilon x - \omega^2 y \quad . \\ a_z &= a_O \end{aligned} \tag{11.32}$$

It follows that the acceleration has the same value, at a given instant, for all the points along a line parallel with the rotation axis, since it contains no $z$ as variable. The modulus of the acceleration increases linearly with increasing distance to the rotation axis (Fig. 11.8).

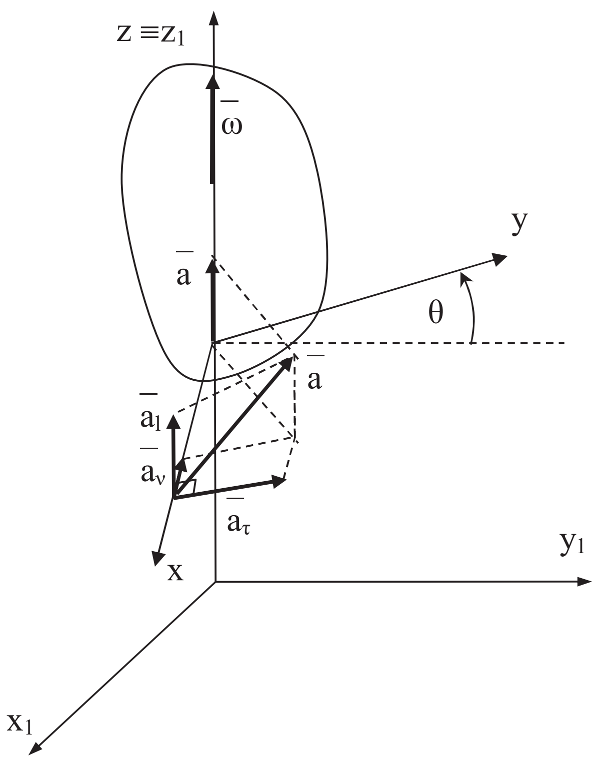



**Fig. 11.8 Acceleration field for the helical motion**

As can be seen from (11.31) the acceleration can be expressed by three components:

- a tangent component

$$\overline{a}_\tau = \varepsilon\left(-y\overline{i} + x\overline{j}\right) \Rightarrow \left|\overline{a}_\tau\right| = \left|\varepsilon\right|\sqrt{x^2 + y^2} = \left|\varepsilon\right| d\,, \tag{11.33}$$

- a normal component

$$\overline{a}_{v} = -\omega^{2}\left(x\overline{i} + y\overline{j}\right) \quad \left|\overline{a}_{v}\right| = \omega^{2}\sqrt{x^{2}+y^{2}} = \omega^{2}d, \tag{11.34}$$

- an axial component

$$\overline{a}_{l} = a_{O}\overline{k} \tag{11.35}$$

in which *d* is again the distance to the rotation axis.

### 11.4.4. Motion of a rigid body parallel to a fixed plane

A rigid body has a motion parallel to a fixed plane (**plane motion**) if there exist three points belonging to the rigid body, which are not situated on the same straight line and which remain in the same fixed plane during motion.

During the motion of a rigid body parallel to a fixed plane, all the points belonging to the rigid body and situated into the plane determined by these three points remain into the fixed plane. The section of the body by the fixed plane is itself a two-dimensional rigid body which will be called **the representative lamina** (Fig. 11.9). It will be proven that the motion of a rigid body parallel to a fixed plane may be reduced to the motion of the representative lamina with respect to the fixed plane. It is possible to choose the axes $Ox_1$ and $Oy_1$ of the fixed Cartesian frame belonging to the fixed plane and the axes $Ox$ and $Oy$ of the movable Cartesian frame into the representative lamina. Obviously $O_1z_1$ and $Oz$ are perpendicular to the fixed plane (and implicitly to the representative lamina) and in general $O \neq O_1$ (Fig. 11.9).

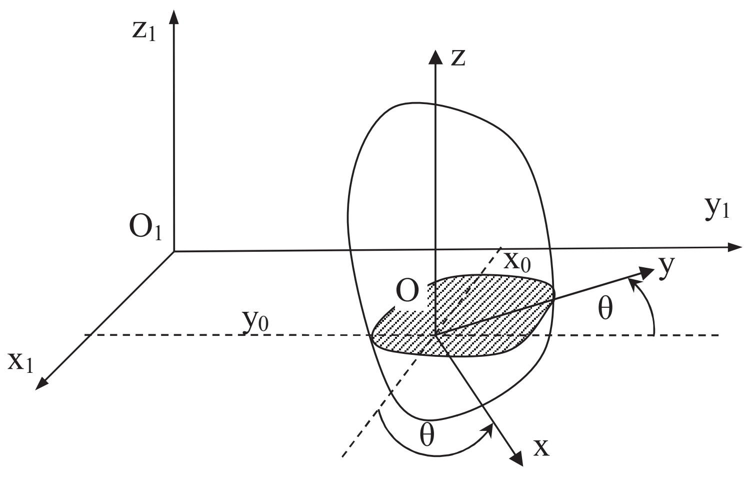



**Fig. 11.9 Plane motion**

Denote by θ the angle between the axes $O_1x_1$ and $Ox$ and by $x_0$ and $y_0$ the coordinates of the origin O of the movable Cartesian frame with respect to the fixed Cartesian frame. The scalar functions θ(t), $x_0(t)$ and $y_0(t)$ define the motion of a rigid body parallel to a fixed plane. It is easy to ascertain that the relative positions of unit vectors $\bar{i}, \bar{j}, \bar{k}$ of the movable Cartesian frame Oxyz with respect to the unit vectors $\bar{i}_1, \bar{j}_1, \bar{k}_1$ of the fixed Cartesian frame are the same as in the case of the rotation. It follows that:

$$\bar{\omega} = \dot{\theta}\bar{k} = \omega\bar{k}; \quad \bar{v}_O = \dot{x}_O\bar{i} + \dot{y}_O\bar{j} = v_{Ox}\bar{i} + v_{Oy}\bar{j} \tag{11.36}$$

*a) Velocity field*

Euler formula becomes

$$\bar{v} = \bar{v}_O + \bar{\omega}\times\bar{r} = v_{Ox}\bar{i} + v_{Oy}j + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \left(v_{Ox} - \omega y\right)\bar{i} + \left(v_{Oy} + \omega x\right)\bar{j} \tag{11.37}$$

The projections of the velocity $\bar{v}$ on the axes of moving frame *Oxyz* are

$$\begin{cases} v_x = v_{Ox} - \omega y \\ v_y = v_{Oy} + \omega x \\ v_z = 0 \end{cases}. \tag{11.38}$$

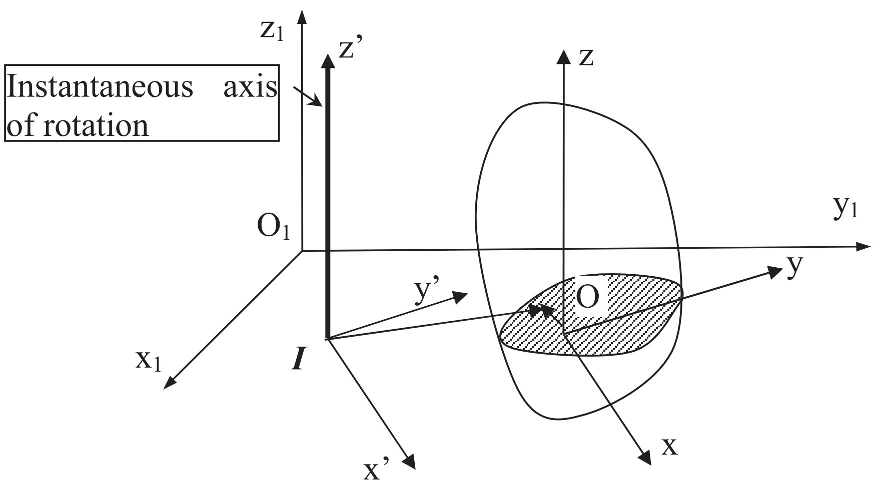



**Fig. 11.10 Instantaneous center and axis of rotation**

It follows that the velocities are constant along a straight line parallel to the axis Oz, i.e. perpendicular to the fixed plane and implicitly to the representative lamina. If $\omega \neq 0$, there is a point ***I*** into the plane *Oxy* for which the velocity is zero. The coordinates of this point are the solutions of the following system of equations:

$$\begin{cases} v_{Ox} - \omega y = 0 \\ v_{Oy} + \omega x = 0 \end{cases}. \tag{11.39}$$

It follows

$$x_I = -\frac{v_{Oy}}{\omega}; \quad y_I = \frac{v_{Ox}}{\omega}. \tag{11.40}$$

A Cartesian frame *Ix'y'z'* with axes parallel to the axes of the moving Cartesian frame *Oxyz* is now considered (Fig. 11.10), the projections of the velocity $\overline{v}$ may be expressed with respect to the new coordinates x', y', z' as follows:

$$\begin{aligned} v_{x'} &= v_{Ox} - \omega\left(\frac{v_{Ox}}{\omega} + y'\right) = -\omega y' \\ v_{y'} &= v_{Oy} + \omega\left(-\frac{v_{Oy}}{\omega} + x'\right) = \omega x' \\ v_{z'} &= 0 \end{aligned} \tag{11.41}$$

Comparing (11.41) with (11.21), it can be concluded that the field of velocity in the motion of a rigid body parallel to a fixed plane is identical with the field of velocity in a rotation as if the rigid body would rotate about an axis perpendicular to the fixed plane and passing through ***I***. This axis is called **instantaneous axis of rotation** and ***I*** is called **instantaneous center of rotation**. Note that the rigid body does not rotate about the instantaneous axis of rotation, because this axis moves and so does the instantaneous centre of rotation.

The geometric locus of the instantaneous center of rotation ***I*** with respect to the fixed Cartesian frame is called the **fixed centrode**. The locus of ***I*** with respect to the movable Cartesian frame is called the **movable centrode**. Obviously the fixed centrode and the movable centrode have at any time *t* a common point, the instantaneous centre of rotation ***I***. It is easy to prove that the movable centrode rolls on the fixed centrode.

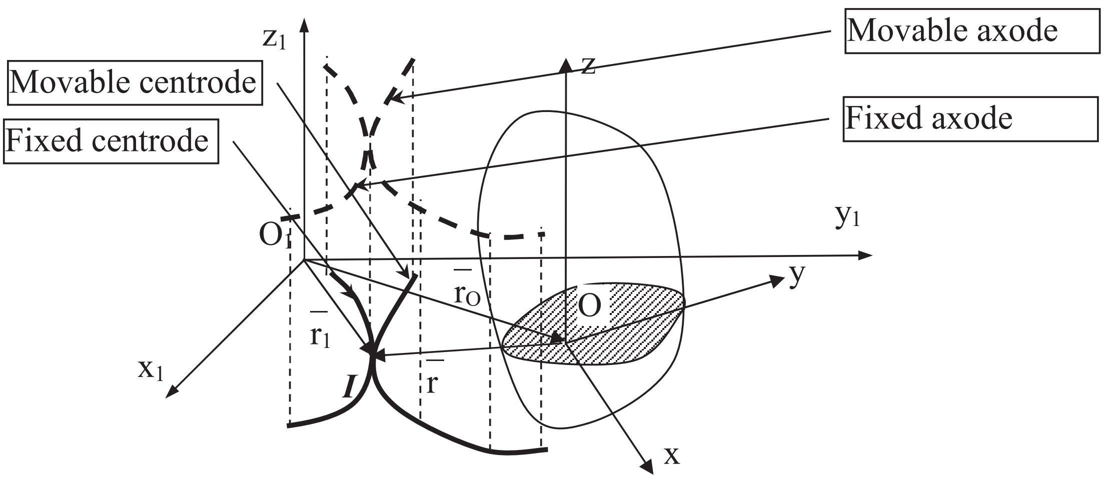



**Fig. 11.11 Rolling of the movable centrode over the fixed centrode**

Indeed, denoting by $x_1$ and $y_1$ the coordinates of ***I*** with respect to the fixed frame $O_1x_1y_1z_1$ and by x and y the coordinates of I with respect to the movable frame Oxyz, the following relation holds (Fig. 11.11):

$$x_1\bar{i}_1 + y_1\bar{j}_1 = \bar{r}_O + x\bar{i} + y\bar{j} \tag{11.42}$$

The derivative of this relation with respect to time is:

$$\dot{x}_1\bar{i}_1 + \dot{y}_1\bar{j}_1 = \dot{\bar{r}}_O + \dot{x}\,\bar{i} + x\,\dot{\bar{i}} + \dot{y}\,\bar{j} + y\,\dot{\bar{j}} \tag{11.43}$$

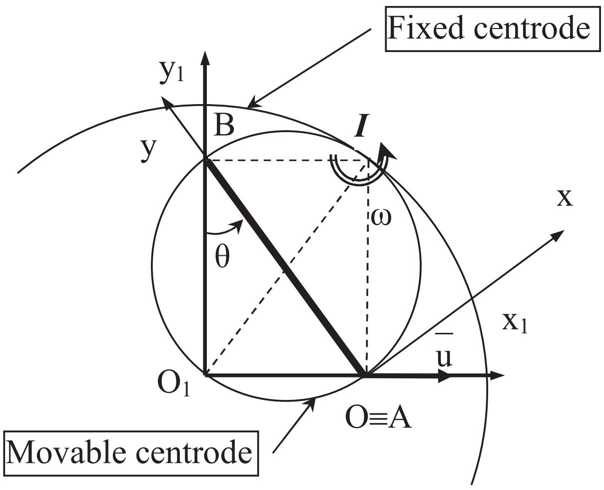
But $\dot{\bar{r}}_O + x\,\dot{\bar{i}} + y\,\dot{\bar{j}} = \bar{v}_O + \bar{\omega}\times\bar{r} = \bar{0}$, because ***I*** is the instantaneous centre of rotation and its velocity is equal to zero. It follows that

$$\dot{x}_1\bar{i}_1 + \dot{y}_1\bar{j}_1 = \dot{x}\,\bar{i} + \dot{y}\,\bar{j} \tag{11.44}$$

The left side of this relation is the velocity of ***I*** with respect to the fixed centrode and the right side of this relation is the velocity of ***I*** with respect to the movable centrode.

The equality of these velocities proves that the two curves have in the common point ***I*** a tangency point. The equality proves also that the elementary arcs of these two curves are equal (because in general $ds = vdt$). It follows that the movable centrode rolls on the fixed centrode. The locus of the instantaneous axis of rotation with respect to the fixed and the movable frame are respectively called the **fixed axode**, and the **movable axode**. These axodes are cylindrical surfaces.

<u>Example (Problem of Cardan)</u>




A rod AB of length $l$ moves in such a way that its ends remain at every instant on the fixed axes $Ox_1$ and $Oy_1$ respectively. The velocity of A is given u > 0 (Fig. 11.12). Determine the instantaneous centre of rotation, the angular velocity, the velocity of B as function of the angle $O_1BA = \theta$, then the fixed centrode and the movable centrode.

**Fig. 11.12 A rod in contact with two fixed orthogonal surfaces**

Since the field of velocity is the same as in rotation, the instantaneous centre ***I*** is at the point of intersection of the straight lines perpendicular to the velocities of A and B. The velocity of A may be written $u = IA\,\omega$. It follows that $\omega = \dfrac{u}{IA} = \dfrac{u}{l\cos\theta}$

The velocity of B may be written $v = IB\,\omega = l\sin\theta \dfrac{u}{\cos\theta} = u\tan\theta$

To determine the fixed centrode, it can be remarked that $O_1I$ = AB = $l$ ($O_1AIB$ is a rectangle) and because $O_1$ is a fixed point, the locus of ***I*** is a circle with center in $O_1$ and radius $l$. The movable centrode: the angle AIB is equal to 90º at every position of the segment AB. The moving centrode will be the circle of the diameter AB = $l$.

It is possible to give also an analytical solution to this problem. Let be $O_1x_1y_1$ and Axy the fixed and respectively the movable Cartesian frames. The coordinates of ***I*** with respect to the fixed and respectively mobile Cartesian frame are denoted by $\zeta_1$ and $\eta_1$ and respectively $\zeta$ and $\eta$:

$$\xi_1 = l\sin\theta; \qquad \eta_1 = l\cos\theta$$

$$\xi = l\cos\theta\sin\theta; \quad \eta = l\cos^2\theta$$

Eliminating the angle $\theta$, it follows $\xi_1^2 + \eta_1^2 = l^2; \quad \xi^2 + \left(\eta - \dfrac{l}{2}\right)^2 = l^2$ , verifying the same results.

*b) Acceleration field*

The Rivals formula, based on the facts:

$$\overline{a}_O = a_{Ox}\overline{i} + a_{Oy}\overline{j}; \quad \overline{\omega} = \omega\overline{k}; \quad \overline{\varepsilon} = \varepsilon\overline{k}\,, \tag{11.45}$$

becomes

$$\overline{a} = \overline{a}_O + \overline{\varepsilon}\times\overline{r} + \overline{\omega}\times\left(\overline{\omega}\times\overline{r}\right) = a_{Ox}\overline{i} + a_{Oy}\overline{j} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & \varepsilon \\ x & y & z \end{vmatrix} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} \tag{11.46}$$
$$= \left(a_{Ox} - \varepsilon y - \omega^2 x\right)\overline{i} + \left(a_{Oy} + \varepsilon x - \omega^2 y\right)\overline{j}$$

The projections of the acceleration a on the axes of moving frame Oxyz are:

$$\begin{aligned} a_x &= a_{Ox} - \varepsilon y - \omega^2 x \\ a_y &= a_{Oy} + \varepsilon x - \omega^2 y\,. \\ a_z &= 0 \end{aligned} \tag{11.47}$$

It follows that the accelerations are constant along a straight line parallel to the axis Oz, which is perpendicular to the fixed plane and implicitly to the representative lamina. If $\omega^4 + \varepsilon^2 \neq 0$ there is a point ***T*** into the representative lamina, for which the acceleration is zero. The coordinates of this point are the solutions of the following system of equations:

$$\begin{aligned} a_{Ox} - \omega^2 x - \varepsilon y = 0 \\ a_{Oy} + \varepsilon x - \omega^2 y = 0 \end{aligned}. \tag{11.48}$$

It follows

$$x_T = \frac{\omega^2 a_{Ox} - \varepsilon a_{Oy}}{\omega^4 + \varepsilon^2}; \qquad y_T = \frac{\omega^2 a_{Oy} + \varepsilon a_{Ox}}{\omega^4 + \varepsilon^2}. \qquad (11.49)$$

The point ***T*** is called the **pole of accelerations**.
As in the case of velocities, if a Cartesian frame Tx"y"z" whose axes are parallel to the axes of the movable frame Oxyz, the projections of the acceleration may be expressed with respect to the new coordinates x", y", z" as follows:

$$a_{x"} = a_{Ox} - \varepsilon\left(\frac{\omega^2 a_{Oy} + \varepsilon a_{Ox}}{\omega^4 + \varepsilon^2} + y"\right) - \omega^2\left(\frac{\omega^2 a_{Oy} + \varepsilon a_{Ox}}{\omega^4 + \varepsilon^2} + x"\right) = -\varepsilon y" - \omega^2 x"$$

$$a_{y"} = a_{Oy} + \varepsilon\left(\frac{\omega^2 a_{Oy} + \varepsilon a_{Ox}}{\omega^4 + \varepsilon^2} + x"\right) - \omega^2\left(\frac{\omega^2 a_{Oy} + \varepsilon a_{Ox}}{\omega^4 + \varepsilon^2} + y"\right) = \varepsilon x" - \omega^2 y" \qquad (11.50)$$

$$a_{z"} = 0$$

Comparing (11.50) to (11.32) it can be concluded that the field of acceleration in the motion of a rigid body parallel to a fixed plane is identical with the field of acceleration in a rotation, as if the rigid body would rotate about an axis perpendicular to the fixed plane and passing through ***T***. Note that the rigid body does not rotate about this axis, because this axis is mobile just as the pole of acceleration T.

<u>Example</u>.
A disc of radius R, whose center O moves with a uniform velocity $\bar{u}$ rolls without sliding along the straight line (Δ = Ox). Determine the instantaneous centre of rotation ***I***, the fixed centrode, and the moving centrode, the pole of accelerations ***T***, the velocity and the acceleration of a point M on the circle (Fig. 11.13).

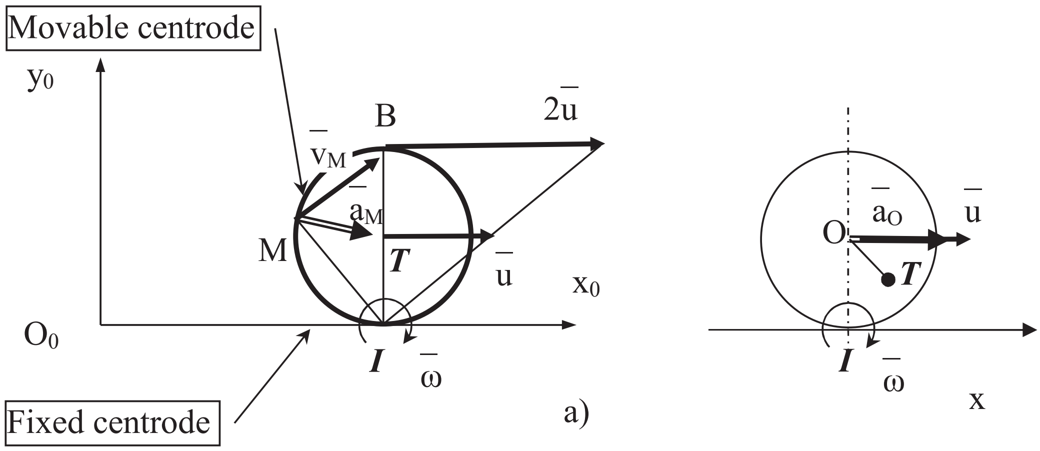



**Fig. 11.13 The motion of "pure" rolling of a disc on a plane surface (a) and pole of accelerations (b)**

Since the circle rolls on the straight line (Δ), the point of contact between the circle and the straight line is the instantaneous centre ***I***. It follows that the fixed centrode is the straight line (Δ) and the movable centrode is the outer circle of the disc. Since the centre O of circle has a uniform rectilinear motion its acceleration is equal to zero. It follows that $\bar{a}_O = \bar{0}$, therefore O is the pole of acceleration (***T***≡O).

Since ***I*** is the instantaneous centre of rotation and the velocity of O is *u*, then $\omega = \frac{u}{R}$. It is very interesting that $v_B = \omega \cdot IB = \frac{u}{R} \cdot 2R = 2u$, so that is if a car moves with a velocity of 50 km/h, then at each instant there are points on the wheels of the train having instantaneous velocities of 100 km/h! For an arbitrary point M on the circle, the velocity $v_M = \omega \cdot IM$ is perpendicular on IM and the acceleration $a_M = TM \cdot \omega^2 = R \cdot \omega^2$ has the direction OM and the sense MO, because $\varepsilon = 0$.

Note. If $a_O \neq 0$ it follows: $\omega = \frac{u}{R} \Rightarrow \varepsilon = \dot{\omega} = \frac{a_O}{R}$. The instantaneous centre of rotation (***I***) is also the point of contact between the circle and the straight line. The pole of accelerations ***T*** however is located as shown in Fig. 11.13b. The segment $OT = \frac{a_O}{\omega^4 + \varepsilon^2}$. The angle $\varphi = IOT$ is given by the relation $\tan\varphi = \frac{\varepsilon}{\omega^2}$.

### 11.4.5. Motion of a rigid body with a fixed point

If a rigid body has a fixed point it is convenient to choose the origins of the fixed and the movable Cartesian frames in this point. It follows that $\bar{v}_O = \bar{0}$ and $\bar{\omega}$ is an arbitrary vector:

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k} . \qquad (11.51)$$

The Euler formula becomes:

$$\bar{v} = \bar{\omega} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (\omega_y z - \omega_z y)\bar{i} + (\omega_z x - \omega_x z)\bar{j} + (\omega_x y - \omega_y x)\bar{k} \qquad (11.52)$$

The projections of the velocity $\bar{v}$ on the axes of movable Cartesian frame are:

$$\begin{aligned} v_x &= \omega_y z - \omega_z x \\ v_y &= \omega_z x - \omega_x z \; . \\ v_z &= \omega_x y - \omega_y x \end{aligned} \qquad (11.53)$$

Obviously, the origin has zero velocity. If other points have zero velocity, the coordinates of these points are solutions of the following system of equations:

$$\begin{aligned} \omega_y z - \omega_z y &= 0 \\ \omega_z x - \omega_x z &= 0 \\ \omega_x y - \omega_y x &= 0 \end{aligned} \qquad (11.54)$$

It follows that:

$$\frac{x}{\omega_x} = \frac{y}{\omega_y} = \frac{z}{\omega_z}. \tag{11.55}$$

This is a straight line passing trough the origin and having the direction of $\overline{\omega}$. This is an **instantaneous axis of rotation**. The field of velocity is identical with that of a rotation as if the rigid body would rotate about this axis. Note that the rigid body does not rotate about this axis because it is a moving one. The locus of the instantaneous axis of rotation with respect to the movable frame is a cone named **polhode cone**.

The locus of the instantaneous axis of rotation with respect to the fixed frame is another cone, named **herpolhode cone**. Obviously these two cones have a common straight line which is the instantaneous axis of rotation. It is easy to prove that these cones are tangent and the polhode cone rolls on the herpolhode cone.

Indeed if $\xi_1$, $\eta_1$, $\zeta_1$ are the coordinates of a certain point P on the instantaneous axis of rotation with respect to the fixed frame $O_1x_1y_1z_1$ and $\xi$, $\eta$, $\zeta$ the coordinates of the same point P with respect to the movable frame Oxyz, the following relation is obvious:

$$\xi_1\bar{i}_1 + \eta_1\bar{j}_1 + \zeta_1\bar{k}_1 = \xi\cdot\bar{i} + \eta\cdot\bar{j} + \zeta\cdot\bar{k} \tag{11.56}$$

The derivative of this relation with respect to the time t, is:

$$\dot{\xi}_1\cdot\bar{i}_1 + \dot{\eta}_1\cdot\bar{j}_1 + \dot{\zeta}_1\cdot\bar{k}_1 = \dot{\xi}\cdot\bar{i} + \dot{\eta}\cdot\bar{j} + \dot{\zeta}\cdot\bar{k} + \xi\cdot\dot{\bar{i}} + \eta\cdot\dot{\bar{j}} + \zeta\cdot\dot{\bar{k}} \tag{11.57}$$

But $\xi\cdot\dot{\bar{i}} + \eta\cdot\dot{\bar{j}} + \zeta\cdot\dot{\bar{k}} = \overline{\omega}\times\overline{r} = 0$ because the considered point P lies on the instantaneous axis of rotation. It follows that:

$$\dot{\xi}_1\cdot\bar{i}_1 + \dot{\eta}_1\cdot\bar{j}_1 + \dot{\zeta}_1\cdot\bar{k}_1 = \dot{\xi}\cdot\bar{i} + \dot{\eta}\cdot\bar{j} + \dot{\zeta}\cdot\bar{k} \tag{11.58}$$

The left side this relation is the velocity of P with respect to the fixed frame and the right side is the velocity of the same point with respect to the movable frame. These two velocities are perpendicular to the instantaneous axis OP because $\xi_1^2 + \eta_1^2 + \zeta_1^2 = \xi^2 + \eta^2 + \zeta^2 = const.$ and consequently by derivation

$$\xi_1\dot{\xi}_1 + \eta_1\dot{\eta}_1 + \zeta_1\dot{\zeta}_1 = \xi\dot{\xi} + \eta\dot{\eta} + \zeta\dot{\zeta} = 0. \tag{11.59}$$

This last formula corresponds to the scalar products between (11.56) and (11.58) which proves the above assertion.

It follows that the two cones have in P the same tangent plane and the movable cone rolls on the fixed cone about the instantaneous axis OP, because the tangent plane at a point P of a cone is determined by the generatrix passing through P and a straight line perpendicular to the generatrix and tangent to the cone. The straight line is in this case the common tangent to the curves described by P on the two cones. The elementary arcs of these two curves are equal, because the two velocities are equal.

The Rivals formula becomes:

$$\bar{a}=\bar{\varepsilon}\times\bar{r}+\bar{\omega}\times(\bar{\omega}\times\bar{r})=\bar{\varepsilon}\times\bar{r}+(\bar{\omega}\cdot\bar{r})\bar{\omega}-\omega^2\bar{r}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \varepsilon_x & \varepsilon_y & \varepsilon_z\\ x & y & z\end{vmatrix} \tag{11.60}$$

$$+(\omega_x x+\omega_y y+\omega_z z)(\omega_x\bar{i}+\omega_y\bar{j}+\omega_z\bar{k})-(\omega_x^2+\omega_y^2+\omega_z^2)(x\bar{i}+y\bar{j}+z\bar{k})$$

The projections of the acceleration $\bar{a}$ on the axes of the movable frame are:

$$\begin{aligned}a_x&=-(\omega_y^2+\omega_z^2)x+(\omega_x\omega_y-\varepsilon_z)y+(\omega_x\omega_z+\varepsilon_y)z\\ a_y&=-(\omega_z^2+\omega_x^2)y+(\omega_y\omega_z-\varepsilon_x)z+(\omega_y\omega_x+\varepsilon_z)x\\ a_z&=-(\omega_x^2+\omega_y^2)z+(\omega_z\omega_x-\varepsilon_y)x+(\omega_z\omega_y+\varepsilon_x)y\end{aligned} \tag{11.61}$$

Obviously, the origin has the acceleration equal to zero. The question is if there are other points with zero acceleration. The coordinates of these points are the solutions of the next system of equations:

$$\begin{aligned}-(\omega_y^2+\omega_z^2)x+(\omega_x\omega_y-\varepsilon_z)y+(\omega_x\omega_z+\varepsilon_y)z&=0\\ -(\omega_z^2+\omega_x^2)y+(\omega_y\omega_z-\varepsilon_x)z+(\omega_y\omega_x+\varepsilon_z)x&=0\\ -(\omega_x^2+\omega_y^2)z+(\omega_z\omega_x-\varepsilon_y)x+(\omega_z\omega_y+\varepsilon_x)y&=0\end{aligned} \tag{11.62}$$

This is a homogeneous linear system. Its determinant can be proven to be:

$$\Delta=-\left[\bar{\omega}\times\bar{\varepsilon}\right]^2=-\left[\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \omega_x & \omega_y & \omega_z\\ \varepsilon_x & \varepsilon_y & \varepsilon_z\end{vmatrix}\right]^2 \tag{11.63}$$

$$=-\left[\left(\omega_y\varepsilon_z-\omega_z\varepsilon_y\right)\bar{i}+\left(\omega_z\varepsilon_x-\omega_x\varepsilon_z\right)\bar{j}+\left(\omega_x\varepsilon_y-\omega_y\varepsilon_x\right)\bar{k}\right]^2$$

If $\bar{\omega}$ and $\bar{\varepsilon}$ are not equal to zero, Δ≠0 and there are no points with zero acceleration, except the origin.
It follows that the field of acceleration of a rigid body with a fixed point is not reducible to the corresponding field in rotation. It is a specific field of acceleration.

Example. Kinematic formulas using Euler angles
A rigid body moves around a fixed point, such that the components of the angular velocity can be expressed using the Euler angles (*Ψ,θ,φ*) as shown on Fig. 11.14, by

$$\bar{\omega}=\dot{\Psi}\,versO_1z_1+\dot{\varphi}\,versOz+\dot{\theta}\,versON \tag{11.64}$$

The nutation angle θ measures the inclination of the instantaneous axis of rotation relative to the fixed $O_1z_1$ axis. The precession angle Ψ is measured in the fixed plane $O_1x_1y_1$ and is positioning the intersection line of the Oxy plane with the mentioned fixed plane. The φ angle is the angle of rotation of the rigid body around the Oz instantaneous axis of rotation. Considering the following relations between the projections of the angular velocity ad the Euler angles, determine the polhode and herpolhode cones assuming the following kinematic conditions $\dot{\Psi} = \omega_0 = const$; $\theta = \theta_0 = const.$ and $\dot{\varphi} = \Omega = const.$. Projecting (11.64) on the axes of the moving frame attached to the rigid body, the angular velocity components are:

$$\begin{aligned} \omega_x &= \dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi \\ \omega_y &= \dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi \\ \omega_z &= \dot{\psi} \cos\theta + \dot{\varphi} \end{aligned} \tag{11.65}$$

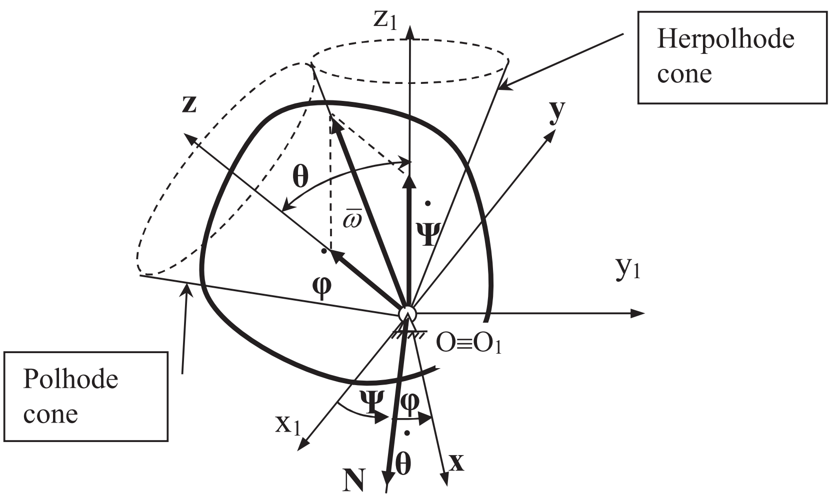



**Fig. 11.14 Motion of a rigid body with a fixed point defined by the Euler angles.**

The angular velocity components are in this particular case:

$$\begin{aligned} \omega_x &= \omega_0 \sin\theta_0 \sin\varphi \\ \omega_y &= \omega_0 \sin\theta_0 \cos\varphi \\ \omega_z &= \omega_0 \cos\theta_0 + \Omega \end{aligned} \tag{11.66}$$

It can be remarked that $\omega_x^2 + \omega_y^2 = \omega_0^2 \sin^2\theta_0 = const.$; $\omega_z = \omega_0 \cos\theta_0 + \Omega = const.$ which means that the angular velocity is a constant and has constant components on the *Oz* axis and on the moving plane *Oxy*. Consequently, the polhode cone has Oz as axis. On the other hand, the projections on the fixed frame of the angular velocities expressed using the Euler angles are:

$$\begin{aligned} \omega_{x1} &= \dot{\theta} \cos\Psi + \dot{\varphi} \sin\theta \sin\Psi = \Omega \sin\theta_0 \sin\Psi \\ \omega_{y1} &= \dot{\theta} \sin\Psi - \dot{\varphi} \sin\theta \cos\Psi = -\Omega \sin\theta_0 \cos\Psi \\ \omega_{z1} &= \dot{\psi} + \dot{\varphi} \cos\theta = \omega_0 + \Omega \cos\theta_0 \end{aligned} \tag{11.67}$$

It follows that the herpolhode cone has the axis on the fixed $O_1z_1$ axis and a constant projection $\Omega\sin\theta_0$ on the fixed $O_1x_1y_1$ plane (Fig. 11.14). This particular motion is called regular precession and will be studied in Dynamics for the rigid body with a fixed point.

### *11.5. General motion of a rigid body*

The Euler formula with arbitrary $\overline{v}_O$ and $\overline{\omega}$ can be written:

$$\overline{v} = \overline{v}_O + \overline{\omega}\times\overline{r} = v_{Ox}\overline{i} + v_{Oy}\overline{j} + v_{Oz}\overline{k} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \left(v_{Ox} + \omega_y z - \omega_z y\right)\overline{i} \qquad (11.68)$$

$$+\left(v_{Oy} + \omega_z x - \omega_x z\right)\overline{j} + \left(v_{Oz} + \omega_x y - \omega_y x\right)\overline{k}$$

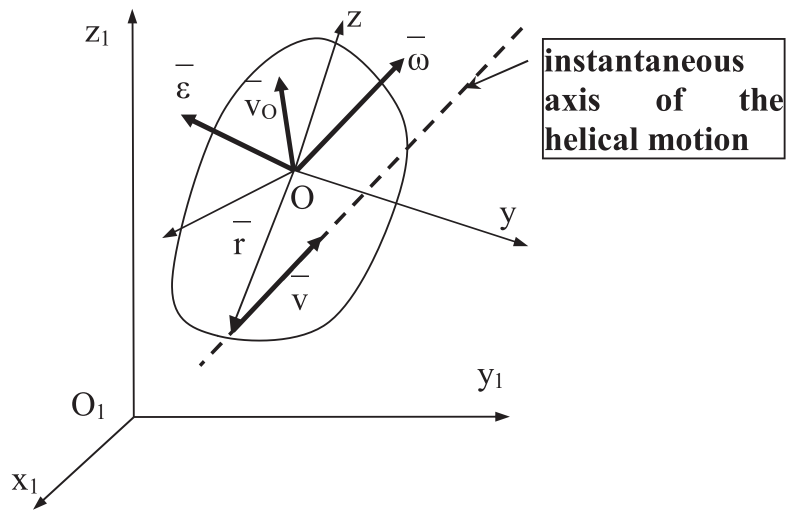



**Fig. 11.15 General motion of a rigid body**

The projections of the velocity $\overline{v}$ of a point of the rigid body on the axes of the movable frame are

$$\begin{aligned} v_x &= v_{Ox} + \omega_y z - \omega_z y \\ v_y &= v_{Oy} + \omega_z x - \omega_x z \\ v_z &= v_{Oz} + \omega_x y - \omega_y x \end{aligned} \qquad (11.69)$$

Property 1: The field of velocity at any time *t* is identical to the field of velocities of a helical motion. Indeed, for the helical motion $\overline{v}_o \parallel \overline{\omega}$, so in this case points for which $\overline{v} \parallel \overline{\omega}$ are those placed on the straight line $\overline{v} = \lambda\overline{\omega}$:

$$\frac{v_{Ox} + \omega_y z - \omega_z y}{\omega_x} = \frac{v_{Oy} + \omega_z x - \omega_x z}{\omega_y} = \frac{v_{Oz} + \omega_x y - \omega_y x}{\omega_z}. \qquad (11.70)$$

This straight line is called the **instantaneous axis of the helical motion** (Fig. 11.15). The general motion of a rigid body may be represented by a succession of infinitesimal helical motions. However the instantaneous axis of the helical motion is mobile. Its locus with respect to the fixed frame is a rectifiable surface called **fixed axode** and its locus with respect to the moving frame is another rectifiable surface called **movable axode**.

Property 2: The two axodes have a common generatrix which is the instantaneous axis of the helical motion.
Property 3: During the motion of the rigid body the movable axode rolls on the fixed axode about the common generatrix and also slides along this generatrix.

Considering $\xi_1, \eta_1, \zeta_1$ the coordinates of a point P belonging to the instantaneous axis of the helical motion with respect to the fixed frame, and $\xi, \eta, \zeta$ the coordinates of the same point but with respect to the moving frame, the following relation holds:

$$\xi_1 \bar{i}_1 + \eta_1 \bar{j}_1 + \zeta_1 \bar{k}_1 = \bar{r}_O + \xi \bar{i} + \eta \bar{j} + \zeta \bar{k}; \ \ \forall t . \tag{11.71}$$

The derivative of this relation with respect to time becomes:

$$\dot{\xi}_1 \cdot \bar{i}_1 + \dot{\eta}_1 \cdot \bar{j}_1 + \dot{\zeta}_1 \cdot \bar{k}_1 = \bar{v}_O + \dot{\xi} \cdot \bar{i} + \dot{\eta} \cdot \bar{j} + \dot{\zeta} \cdot \bar{k} + \xi \cdot \dot{\bar{i}} + \eta \cdot \dot{\bar{j}} + \zeta \cdot \dot{\bar{k}} . \tag{11.72}$$

But $\bar{v}_O + \xi \cdot \dot{\bar{i}} + \eta \cdot \dot{\bar{j}} + \zeta \cdot \dot{\bar{k}} = \bar{v}_O + \bar{\omega} \times \bar{r} = \lambda \bar{\omega}$ because the point P is on the instantaneous axis of the helical motion and its velocity is parallel to the angular velocity. It follows:

$$\dot{\xi}_1 \cdot \bar{i}_1 + \dot{\eta}_1 \cdot \bar{j}_1 + \dot{\zeta}_1 \cdot \bar{k}_1 = \dot{\xi} \cdot \bar{i} + \dot{\eta} \cdot \bar{j} + \dot{\zeta} \cdot \bar{k} + \lambda \bar{\omega} . \tag{11.73}$$

Since the difference between the velocity of the point *P* with respect to the fixed frame and the velocity of *P* with respect to the moving frame has the direction of $\bar{\omega}$, it follows that the movable axode is sliding along the common generatrix and rolls about this generatrix.

The acceleration of an arbitrary point is given by the Rivals formula:

$$\bar{a} = \bar{a}_O + \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times \left( \bar{\omega} \times \bar{r} \right), \tag{11.74}$$

with arbitrary $\bar{a}_O$, $\bar{\omega}$, $\bar{\varepsilon}$ .

Property 4: The field of accelerations in a general motion of a rigid body is identical to the field of accelerations in a motion of a rigid body with a fixed point.

If $\bar{\omega}$ and $\bar{\varepsilon}$ are not equal to zero, there is a point and only a point of position vector $\bar{r}$ for which the acceleration $\bar{a} = \bar{a}_O + \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times \left( \bar{\omega} \times \bar{r} \right) = \bar{0}$ . This condition can be expressed as:

$$\bar{\varepsilon} \times \bar{r} + \bar{\omega} \times \left( \bar{\omega} \times \bar{r} \right) = -\bar{a}_O . \qquad (11.75)$$

Scalar multiplying by $\bar{\omega}$, leads to:

$$\bar{\omega}\left( \bar{\varepsilon} \times \bar{r} \right) + \bar{\omega}\left[ \bar{\omega} \times \left( \bar{\omega} \times \bar{r} \right) \right] = -\bar{\omega} \cdot \bar{a}_O \qquad (11.76)$$

The second term in the left side is null being a mixed product with two identical vectors. It follows:

$$\left( \bar{\omega} \times \bar{\varepsilon} \right) \bar{r} = -\bar{\omega} \cdot \bar{a}_O . \qquad (11.77)$$

In general the angular velocity and acceleration are not collinear ($\bar{\omega} \times \bar{\varepsilon} \neq \bar{0}$), so that a unique non-trivial solution for $\bar{r}$ exists.
The coordinates of this point are the solutions of the following system of algebraic equations, obtained from (11.75):

$$\begin{aligned} -(\omega_y^2 + \omega_z^2)x + (\omega_x \omega_y - \varepsilon_z)y + (\omega_x \omega_z + \varepsilon_y)z &= -a_{Ox} \\ -(\omega_z^2 + \omega_x^2)y + (\omega_y \omega_z - \varepsilon_x)z + (\omega_y \omega_x + \varepsilon_z)x &= -a_{Oy} \\ -(\omega_x^2 + \omega_y^2)z + (\omega_z \omega_x - \varepsilon_y)x + (\omega_z \omega_y + \varepsilon_x)y &= -a_{Oz} \end{aligned} \qquad (11.78)$$

# 12. KINEMATICS OF THE RELATIVE MOTION

## 12.1. The relative motion of a point. Preliminaries

The velocity and acceleration of a point depend on the reference frame, relative to which the motion of the point is examined. The motion of one and the same point will therefore be described differently by two observers moving relative to each other. The motions of the planets and Sun relative to a reference frame attached to the Earth are very complicated. Copernicus discovered that the motions of the planets are represented in a much simpler manner, if the reference frame is attached to the Sun.

The problem is thus to determine the motion of a point M relative to one ”fixed” frame as this motion is determined relative to another moving frame. The motion of M with respect to the moving frame is by definition the relative motion of M. The motion of M with respect to the “fixed” frame is by definition the **absolute motion**. The **motion of transport** at a given moment is the motion of a point belonging to the moving frame and coinciding at the given moment with the point M. It is possible to define the **relative path**, the **relative velocity**, the **relative acceleration**, the **absolute path**, the **absolute velocity**, the **absolute acceleration**, but only the **velocity** and the **acceleration of transport**.

Suppose for example, that a passenger is running along the aisle of train. As the fixed frame, can be taken the frame attached to the Earth and as moving frame, the frame attached to the train. A person standing near the track will observe the motion of the passenger relative to the fixed frame (the absolute motion) and a person sitting in the train car will observe the motion in the moving frame (the relative motion). The velocity of transport and the acceleration of transport will be the velocity and the acceleration of that point of the floor belonging to the aisle on which the running person is at a given moment.

In the following, it shall be determined the absolute velocity and the absolute acceleration, if the relative motion of the point M with respect to the moving frame and the motion of the moving frame with respect to the fixed frame are known.

## 12.2. The derivative of a vector defined by projections on the axes of a moving frame

Let $\overline{U}$ be a vector whose projections on a moving frame are known: $\overline{U} = U_x\overline{i} + U_y\overline{j} + U_z\overline{k}$ . The derivative of this relation with respect to time can be written successively:

$$\frac{d\overline{U}}{dt} = \dot{U}_x\overline{i} + \dot{U}_y\overline{j} + \dot{U}_z\overline{k} + U_x\dot{\overline{i}} + U_y\dot{\overline{j}} + U_z\dot{\overline{k}} \,. \qquad (12.1)$$

By virtue of the Poisson formula:

$$\begin{aligned} U_x\dot{\bar{i}} + U_y\dot{\bar{j}} + U_z\dot{\bar{k}} &= U_x\bar{\omega}\times\bar{i} + U_y\bar{\omega}\times\bar{j} + U_z\bar{\omega}\times\bar{k} \\ &= \bar{\omega}\times\left(U_x\bar{i} + U_y\bar{j} + U_z\bar{k}\right) = \bar{\omega}\times\bar{U} \end{aligned} \tag{12.2}$$

The **local derivative** of $\bar{U}$ with respect to time is by definition:

$$\frac{\partial\bar{U}}{\partial t} = \dot{U}_x\bar{i} + \dot{U}_y\bar{j} + \dot{U}_z\bar{k}\,. \tag{12.3}$$

The expression (12.1) becomes

$$\frac{d\bar{U}}{dt} = \frac{\partial\bar{U}}{\partial t} + \bar{\omega}\times\bar{U}\,. \tag{12.4}$$

### *12.3. Absolute velocity of a point*

A fixed Cartesian frame $O_1x_1y_1z_1$ , a moving Cartesian frame *Oxyz* and a moving point M are considered (Fig. 12.1).

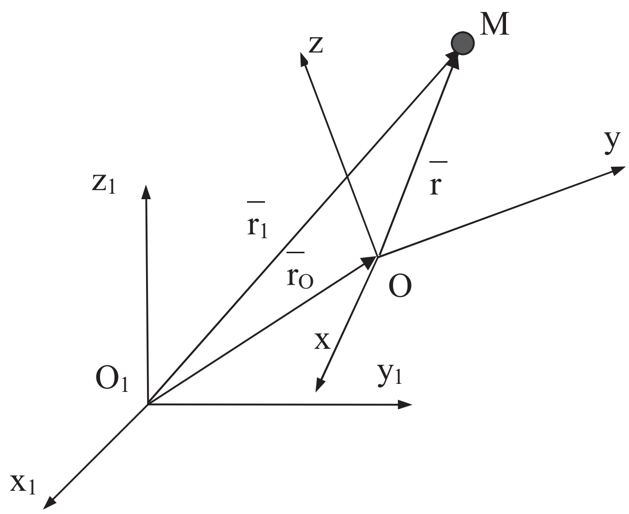



**Fig. 12.1 Relative and absolute position vectors**

The following notations are used:

$\bar{r} = \overline{OM}$ the position vector of M with respect to the moving frame

$\bar{r}_1 = \overline{O_1M}$ the position vector of M with respect to the fixed frame and

$\bar{r}_0 = \overline{O_1O}$ the position vector of the origin O of the moving frame with respect to the fixed frame. Obviously:

$$\bar{r}_1 = \bar{r}_0 + \bar{r}\,. \tag{12.5}$$

The derivative of this relation with respect to the time becomes:

$$\dot{\bar{r}}_1 = \dot{\bar{r}}_O + \frac{\partial \bar{r}}{\partial t} + \bar{\omega} \times \bar{r}\,, \tag{12.6}$$

because $\bar{r}$ is defined by its projections on the axes of the moving frame and its time derivative is obtained by applying (12.4).

Since $\dot{\bar{r}}_1$ is the absolute velocity of M, $\dot{\bar{r}}_O = \bar{v}_O$ is the velocity of O and $\frac{\partial \bar{r}}{\partial t} = \bar{v}_r$ is the relative velocity of M, the relation (12.6) may be written:

$$\bar{v}_a = \bar{v}_r + \bar{v}_0 + \bar{\omega} \times \bar{r}\,. \tag{12.7}$$

The velocity of transport is the velocity of that point of the moving frame having the position vector $\bar{r}$:

$$\bar{v}_t = \bar{v}_0 + \bar{\omega} \times \bar{r}\,. \tag{12.8}$$

The relation (12.7) becomes

$$\bar{v}_a = \bar{v}_r + \bar{v}_t \tag{12.9}$$

Therefore the **absolute velocity** is equal to the sum of the **relative velocity** and the **velocity of transport**.

### *12.4. Absolute acceleration of a point*

The time derivative of relation (12.6) is:

$$\ddot{\bar{r}}_1 = \ddot{\bar{r}}_O + \frac{\partial^2 \bar{r}}{\partial t^2} + \bar{\omega} \times \frac{\partial \bar{r}}{\partial t} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \left( \frac{\partial \bar{r}}{\partial t} + \bar{\omega} \times \bar{r} \right), \tag{12.10}$$

because $\bar{r}$ and $\frac{\partial \bar{r}}{\partial t}$ are defined by their projections on the axes of the moving frame and their derivatives are obtained by applying formula (12.4). Since $\ddot{\bar{r}}_1$ is the absolute acceleration of M, $\ddot{\bar{r}}_O$ is the acceleration of O, $\frac{\partial \bar{r}}{\partial t} = \bar{v}_r$ is the relative velocity and $\frac{\partial^2 \bar{r}}{\partial t^2} = \bar{a}_r$ is the relative acceleration, then the relation (12.10) may be written:

$$\bar{a}_a = \bar{a}_r + \bar{a}_0 + \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times (\bar{\omega} + \bar{r}) + 2\bar{\omega} \times \bar{v}_r \tag{12.11}$$

The acceleration of transport is the acceleration of that point of the moving frame having the position vector $\bar{r}$ and is obtainable from Rivals formula:

$$\bar{a}_t = \bar{a}_0 + \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})\,. \tag{12.12}$$

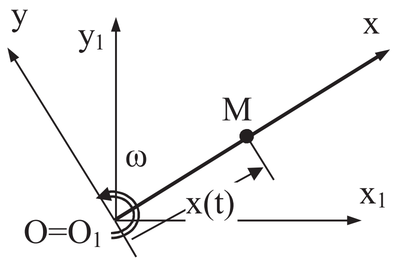
The **Coriolis acceleration** is by definition twice the vector product of the angular velocity of the transport velocity and the relative velocity of the point:

$$\bar{a}_c = 2\bar{\omega} \times \bar{v}_r \,. \tag{12.13}$$

The relation (12.10) becomes:

$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_c \,. \tag{12.14}$$

Hence the absolute acceleration is equal to the sum of the relative acceleration, the acceleration of transport and the Coriolis acceleration.




Example 1. A point M moves along a bar situated on the Ox axis of a moving Cartesian frame Oxyz (Fig. 12.2). The moving Cartesian frame Oxyz rotates about the axis $O_1 z_1 \equiv Oz$ with angular velocity $\bar{\omega}$. Determine the absolute velocity and the absolute acceleration of M for $x = \frac{1}{2}\gamma t^2$, if $\omega, \gamma$ are positive constants.

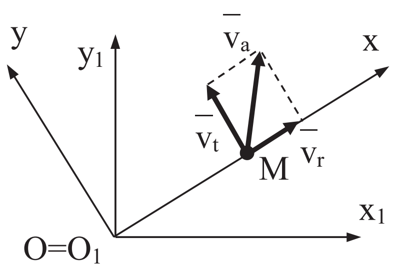
**Fig. 12.2 A point moving on a rotating bar**

From the given data, it follows that $\bar{r}_0 = 0$ ; $\bar{r} = x\bar{i} = 1/2\gamma t^2 \bar{i}$; $\bar{\omega} = \omega \bar{k}$, $\bar{\varepsilon} = 0$, $\bar{v}_O = \dot{\bar{r}}_O = \bar{0}$; $\bar{a}_O = \dot{\bar{v}}_O = \bar{0}$.

In order to obtain the absolute velocity of M, the relative velocity and velocity of transport are first determined:

$$\bar{v}_r = \frac{\partial \bar{r}}{\partial t} = \gamma t \bar{i};$$

$$\bar{v}_t = \bar{v}_0 + \bar{\omega} \times \bar{r} = \omega \bar{k} \times 1/2\gamma t^2 \bar{i} = 1/2\gamma\omega t^2 \bar{j}$$

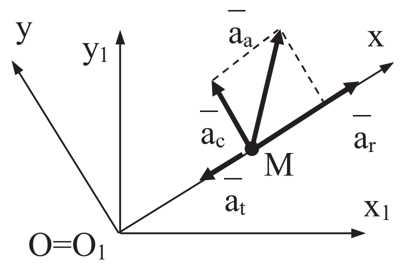



It follows that the absolute velocity of M is:

$$\bar{v}_a = \bar{v}_r + \bar{v}_t = \gamma t \bar{i} + 1/2\gamma\omega t^2 \bar{j}$$

For the absolute acceleration of M, its relative acceleration, acceleration of transport and Coriolis acceleration are separately determined:

$$\bar{a}_r = \frac{\partial^2 \bar{r}}{\partial t^2} = \gamma \bar{i}$$



$$\overline{a}_t = \overline{a}_0 + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r}) =$$

$$= \omega \overline{k} \times \frac{\gamma}{2} \omega t^2 \overline{j} = -\frac{\gamma}{2} \omega^2 t^2 \overline{i}$$

$$\overline{a}_c = 2\overline{\omega} \times \overline{v}_r = 2\omega \overline{k} \times \gamma t \overline{i} = 2\omega \gamma t \overline{j} \ .$$

It follows that the absolute acceleration of M is:

$$\overline{a}_a = \overline{a}_r + \overline{a}_t + \overline{a}_c = (\gamma - \frac{\gamma}{2} \omega^2 t^2) \overline{i} + 2\gamma \omega t \overline{j}$$

Example 2. Determine by a graphical method the relative velocity, the velocity of transport of A and the relative acceleration, acceleration of transport for the present position of a mechanism shown on Fig. 12.3. The constant angular velocity of *OA* is $\omega$ and *OA=r*.

The absolute velocity of A is $|\overline{v}_A| = \omega r = \omega OA$. It is perpendicular to OA. The direction of the relative velocity of A is $O_1 A$ and the direction of the transport velocity of A is perpendicular to $O_1 A$. At a chosen scale, with the origin in an arbitrary point P, can be drawn the vector $\overline{v}_a$ in the sense given by $\omega$ (see figure b). From geometrical properties of triangles, it follows that:

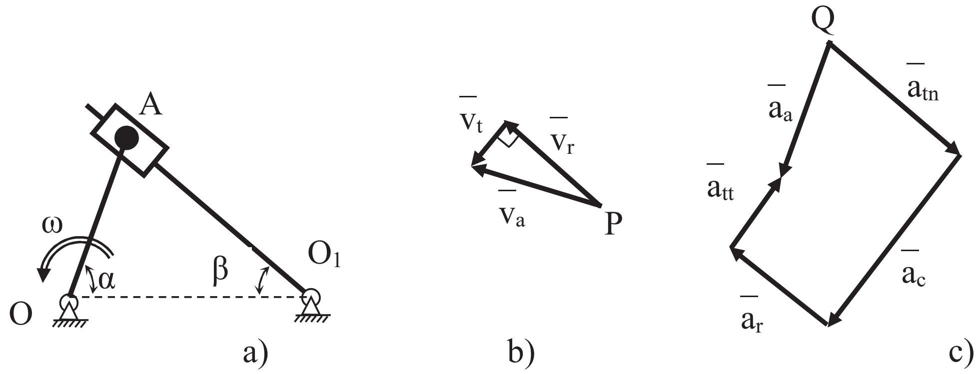



**Fig. 12.3 A slider mechanism (a), the velocities diagram (b), the accelerations diagram(c)**

$|\overline{v}_r| = v_a \sin(\alpha + \beta) = r\omega \sin(\alpha + \beta)$; $|\overline{v}_t| = v_A \cos(\alpha + \beta)$. The angular velocity of the transport motion is $\omega_t = \frac{v_t}{O_1 A}$.

The absolute acceleration of A is $|\overline{a}_A| = OA\omega^2 = r\omega^2$. Its direction and sense are given by $\overline{AO}$. The normal acceleration of transport and the Coriolis acceleration of A are respectively:

$|\overline{a}_{tn}| = \frac{v_t^2}{O_1 A} = O_1 A \omega_t^2$ (Its direction and sense are given by $\overline{AO_1}$) and:

$|a_c| = 2\omega_t v_r = 2\frac{v_t}{O_1 A} v_r = 2\frac{r^2 \omega^2}{O_1 A} \sin(\alpha + \omega) \cos(\alpha + \omega)$ (Perpendicular to $O_1 A$)

From an arbitrary point Q are plotted $\overline{a}_a$ and one after another: $\overline{a}_{tn}$, $\overline{a}_c$.

The direction of $\overline{a}_r$ is $O_1A$ and the direction of the tangent acceleration of transport $\overline{a}_{tt}$ is perpendicular to $O_1A$. These last two vectors must close the polygon of vectors. Their values result from the very polygon of vectors (fig c)
$|\overline{a}_r| = |\overline{a}_{tn}| - |\overline{a}_a|\cos(\alpha+\beta)$; $|\overline{a}_{tt}| = |\overline{a}_c| - |\overline{a}_a|\sin(\alpha+\beta)$.

## *12.5. Relative motion of a rigid body. Relations for velocities*

A rigid body is moving with respect to a movable frame $O_1x_1y_1z_1$. A fixed Cartesian frame *Oxyz* and a Cartesian frame $O_2x_2y_2z_2$ attached to the rigid body are also considered. Denoted are $\overline{O_1M} = \overline{r}_1$ and $\overline{O_2M} = \overline{r}_2$ the position vectors of a certain point M of the rigid body with respect to the origins of two Cartesian frames (Fig. 12.4).

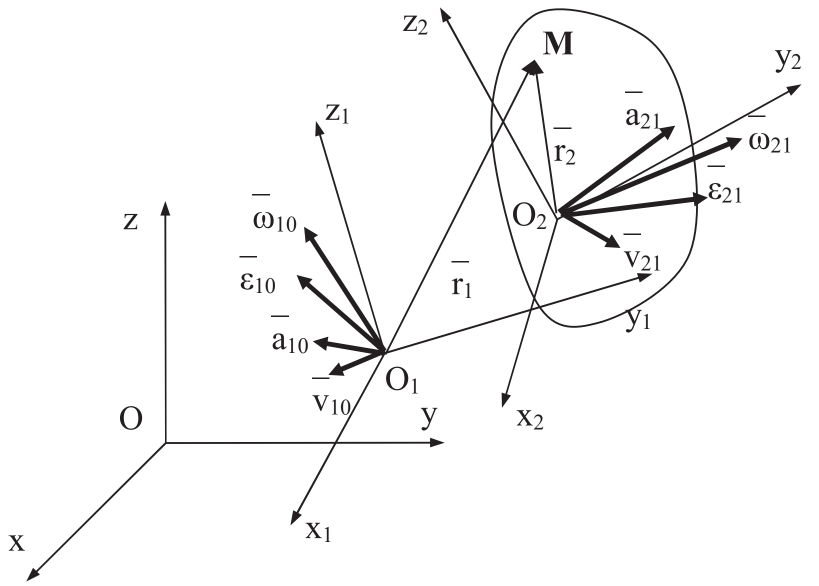



**Fig. 12.4 Relative motion of a rigid body**

The relative angular velocities and accelerations are $\overline{\omega}_{10}, \overline{\omega}_{21}, \overline{\varepsilon}_{10}, \overline{\varepsilon}_{21}$. The relative velocities and accelerations are $\overline{v}_{10}, \overline{v}_{21}, \overline{a}_{10}, \overline{a}_{21}$. These vectors define the motions of the origins $O_1$ and $O_2$ and also defining the motions of $O_1x_1y_1z_1$ with respect to *Oxyz* and $O_2x_2y_2z_2$ with respect to $O_1x_1y_1z_1$ respectively.
In order to determinate the velocity of arbitrary point M with respect to the fixed Cartesian frame Oxyz, it will be considered the velocity of M with respect to $O_1x_1y_1z_1$ as a "relative velocity" and the velocity of the point belonging to the $O_1x_1y_1z_1$ frame and coinciding with M as a "velocity of transport". It follows that:

$$\overline{v}_r = \overline{v}_{21} + \overline{\omega}_{21} \times \overline{r}_2$$
$$\overline{v}_t = \overline{v}_{10} + \overline{\omega}_{10} \times \overline{r}_1 \qquad (12.15)$$

Consequently:

$$\bar{v}_M = \bar{v}_{10} + \bar{v}_{21} + \bar{\omega}_{10} \times \bar{r} + \bar{\omega}_{21} \times \bar{r}_{21} \quad (12.16)$$

It is easy to generalize this formula for $n$ movable Cartesian frames $O_1x_1y_1z_1 ... O_nx_ny_nz_n$. The absolute velocity is in general:

$$\bar{v}_M = \sum_{i=1}^{n} \bar{v}_{i,i-1} + \sum_{i=1}^{n} \bar{\omega}_{i,i-1} \times \bar{r}_i \quad (12.17)$$

Another point N belonging to the rigid body is now considered. If $\bar{r}_j$ ($j=1...n$) are the position vectors of N with respect to the origins of the movable Cartesian frames $O_1x_1y_1z_1 ... O_nx_ny_nz_n$ the expression of $\bar{v}_N$ is:

$$\bar{v}_N = \sum_{i=1}^{n} \bar{v}_{i,i-1} + \sum_{i=1}^{n} \bar{\omega}_{i,i-1} \times \bar{r}_j \quad (12.18)$$

By subtracting the relation (12.18) from (12.17) it is obtained:

$$\bar{v}_M - \bar{v}_N = \sum_{i=1}^{n} \bar{\omega}_{i,i-1} \times (\bar{r}_i - \bar{r}_j) = \sum_{i=1}^{n} \bar{\omega}_{i.i-1} \times \overline{MN} \quad (12.19)$$

because $\bar{r}_j - \bar{r}_i = \overline{MN}$. The last relation may be written:

$$\bar{v}_N = \bar{v}_M + (\sum_{i=1}^{n} \bar{\omega}_{i.i-1}) \times \overline{MN} \quad (12.20)$$

Since the velocity field of a rigid body $n$ is defined by the relation

$$\bar{v} = \bar{v}_0 + \bar{\omega} \times \bar{r} \quad (12.21)$$

it follows if the origin O is taken in M, that $\bar{v} = \bar{v}_N$, $\bar{r} = \overline{MN}$, $\bar{\omega} = \bar{\omega}_{n0}$ and

$$\bar{v}_N = \bar{v}_M + \bar{\omega}_{no} \times \overline{MN} \quad (12.22)$$

By comparing (12.22) and (12.20) it follows that:

$$\bar{\omega}_{n0} = \sum_{i=1}^{n} \bar{\omega}_{i.i-1} \quad (12.23)$$

## *12.6. The Kinematic-Static analogy*

The formulas (12.17) and (12.23) may be written in the form:

$$\bar{\omega}_0 = \sum_{i=1}^{n} \bar{\omega}_{i.i-1}; \quad \bar{v}_M = \sum_{i=1}^{n} \overline{MO_i} \times \bar{\omega}_{i,i-1} + \sum_{i=1}^{n} \bar{v}_{i,i-1}, \quad (12.24)$$

because $\bar{r}_i = \overline{O_iM} = -\overline{MO_i}$ and $\bar{\omega}_{i,i-1} \times \bar{r}_i = \bar{\omega}_{i,i-1} \times \overline{O_iM} = \overline{MO_i} \times \bar{\omega}_{i,i-1}$.
These formulas are analogous to the formulas of Statics:

$$\overline{R} = \sum_{i=1}^{n} \overline{F}_i; \quad \overline{M}_0 = \sum_{i=1}^{n} \overline{MO_i} \times \overline{F}_i + \sum_{i=1}^{n} \overline{M}_i \, , \tag{12.25}$$

for the resultant force vector $\overline{R}$ and for the resultant moment vector $\overline{M}_0$ of a system of forces $\overline{F}_i$ and couples $\overline{M}_i$ acting upon a rigid body. It follows from these the character of sliding vectors for $\overline{\omega}_i$ and the character of free vectors for $\overline{v}_i$. It follows also an analogy:

$$\overline{\omega} \longleftrightarrow \overline{F}; \quad \overline{v} \longleftrightarrow \overline{M} \tag{12.26}$$

This analogy is called the kinematical-static analogy.

## 12.7. Velocity field of the absolute motion

The kinematic-static analogy permits to find the velocity field of the resultant motion of *n* relative motions of moving frames $O_1x_1y_1z_1 \ldots O_nx_ny_nz_n$. Let $\overline{\omega}_{n0}$ and $\overline{v}_M$ by the absolute angular velocity and the velocity of a certain point M of a rigid body. The following possible cases exist:

1) $\overline{\omega}_{n0} = \overline{0}$, $\overline{v}_M = \overline{0}$ then the rigid body is at rest
2) $\overline{\omega}_{n0} \neq \overline{0}$, $\overline{v}_M = \overline{0}$ then the velocity field is as in rotation
3) $\overline{\omega}_{n0} = \overline{0}$, $\overline{v}_M \neq \overline{0}$ then the velocity field is as in translation
4) $\overline{\omega}_{n0} \neq \overline{0}$ $\overline{v}_M \neq \overline{0}$ and
   a) $\overline{\omega}_{n0} \cdot \overline{v}_M = 0$ then the velocity field is as in rotation
   b) $\overline{\omega}_{n0} \cdot \overline{v}_M \neq 0$ then the velocity field is as in helical motion

## 12.8. Superposition of particular motions of rigid bodies. Velocity problem

### 12.8.1. Superposition of translations

In this case $\overline{\omega}_{i,i-1} = 0$; $\overline{v}_{i,i-1} \neq 0$ ($i$=*1*...*n*).It follows that:

$$\begin{aligned} \overline{\omega}_{n0} &= \sum_{i=1}^{n} \overline{\omega}_{i,i-1} = \overline{0} \\ \overline{v}_M &= \sum_{i=1}^{n} \overline{MO_i} \times \overline{\omega}_{i,i-1} + \sum_{i=1}^{n} \overline{v}_{i.i-1} = \sum_{i=1}^{n} \overline{v}_{i,i-1} \end{aligned} \tag{12.27}$$

Two cases are possible:

a) if $\overline{v}_M \neq \overline{0}$ the velocity field of the resultant motion is a translation,

b) if $\overline{v}_M = \overline{0}$ the rigid body is at rest.

### 12.8.2. Superposition of concurrent rotations

In this case: $\overline{\omega}_{i,i-1} \neq \overline{0}$ and $\overline{v}_{i,i-1} = \overline{0}$ and it is assumed that $O_i \equiv O$ ($i=1....n$).
It follows that

$$\overline{\omega}_{n0} = \sum_{i=1}^{n} \overline{\omega}_{i,i-1}$$
$$\overline{v}_M = \sum_{i=1}^{n} \overline{MO} \times \overline{\omega}_{i,i-1} + \sum_{i=1}^{n} \overline{v}_{i,i-1} = \overline{MO} \times \sum_{i=1}^{n} \overline{\omega}_{i,i-1} = \overline{MO} \times \overline{\omega}_{no} \tag{12.28}$$

Two cases are possible:

a) if $\overline{\omega}_{n0} \neq \overline{0}$ the velocity filed of the resultant motion is a rotation

b) if $\overline{\omega}_{n0} = \overline{0}$ the rigid body is at rest.

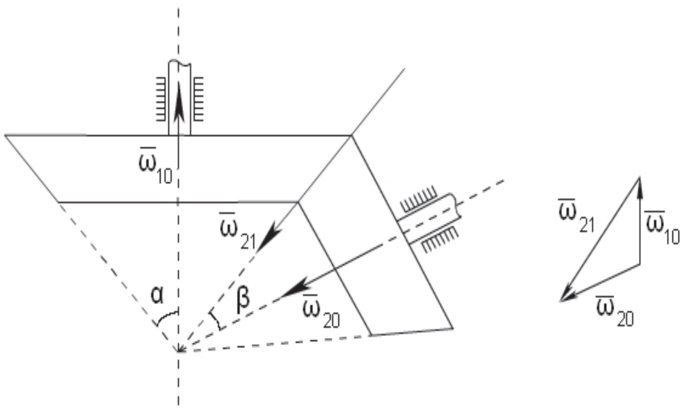



**Fig. 12.5 Two conical gears**

<u>Example</u>

Two conical gears having concurrent fixed axes and the angles between axes and generatrices α and β respectively have at each time a common generatrix (Fig. 12.5). If $\overline{\omega}_{10}$ is the angular velocity of the first gear, determine the absolute angular velocity $\overline{\omega}_{20}$ of the second gear and the relative angular velocity $\overline{\omega}_{21}$. From $\overline{\omega}_{20} = \overline{\omega}_{10} + \overline{\omega}_{21}$ it follows that these three vectors form a triangle (see figure). Applying the law of sinus in this triangle, it is obtained:

$$\frac{\omega_{10}}{\sin\beta} = \frac{\omega_{21}}{\sin(\beta+\alpha)} = \frac{\omega_{20}}{\sin\alpha} \text{ and finally : } \omega_{21} = \frac{\sin(\alpha+\beta)}{\sin\beta}\omega_{10} \quad \omega_{20} = \frac{\sin\alpha}{\sin\beta}\omega_{10}$$

### 12.8.3. Superposition of parallel rotations

In this case: $\overline{\omega}_{i,i-1} = \omega_{i,i-1}\overline{u}$ and $\overline{v}_{i,i-1} = \overline{0}$ ($i=1,...,n$). It follows that

$$\overline{\omega}_{no} = \sum_{i=1}^{n} \overline{\omega}_{i,i-1} = \left(\sum_{i=1}^{n} \omega_{i,i-1}\right)\overline{u}$$
$$\overline{v}_M = \sum_{i=1}^{n} \overline{MO_i} \times \overline{\omega}_{i,i-1} + \sum_{i=1}^{n} \overline{v}_{i,i-1} = \left(\sum_{i=1}^{n} \overline{MO_i} \cdot \omega_{i,i-1}\right) \times \overline{u} \tag{12.29}$$

Three cases are possible:

a) if $\overline{\omega}_{no} = \overline{0}$ and $\overline{v}_M = \overline{0}$ the rigid body is at rest;

b) if $\overline{\omega}_{no} = \overline{0}$ and $\overline{v}_M \neq \overline{0}$ the velocity field of the resultant motion is a translation

c) if $\overline{\omega}_{no} \neq \overline{0}$ the velocity field of the resultant motion is a rotation.

By analogy with the case of parallel forces in this later case the instantaneous axis of rotation passes through the center of parallel vectors $\overline{\omega}_{i,i-1}$ whose position vector is:

$$\overline{\rho} = \frac{\sum \overline{MO_i} \cdot \omega_{i,i-1}}{\sum \omega_{i,i-1}} \tag{12.30}$$

<u>Example</u>

Two cylindrical gears have parallel fixed axes and the radii $r_1$ and $r_2$. They have at each time a common generatrix (Fig. 12.6). If $\overline{\omega}_{10}$ is the angular velocity of the first gear, determine the absolute angular velocity $\overline{\omega}_{20}$ of the second gear and the relative angular velocity $\overline{\omega}_{21}$.

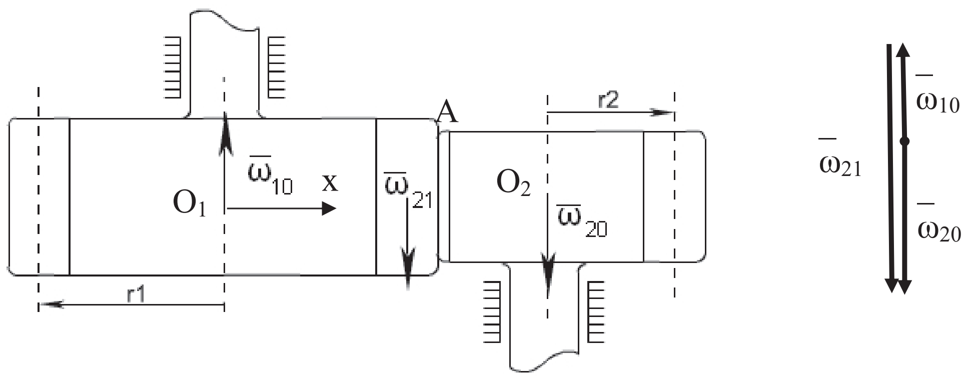



**Fig. 12.6 Two parallel gears**

From the equations (12.29):

$$\overline{\omega}_{20} = \overline{\omega}_{10} + \overline{\omega}_{21}$$

and expressing the velocity of a point $O_2$ on the axis of the second gear:

$$\overline{0} = \left(-O_2O_1 \cdot \omega_{10} - O_2A \cdot \omega_{21}\right)\overline{i} = -\left(r_1 + r_2\right)\overline{i}\,\omega_{10} - r_2\overline{i}\,\omega_{21}$$

it follows that

$$\omega_{20} = -\frac{r_1}{r_2}\omega_{10} \;;\; \omega_{21} = -\frac{r_1 + r_2}{r_2}\omega_{10}$$

The radiuses $r_1$ and $r_2$ are measured at pitch circles and pitch is equal on both gears. It follows a direct proportionality between these radiuses and the number of teeth $z_1$ and $z_2$ of the two gears. Consequently the reduction factor is:

$$\left|\frac{\omega_{20}}{\omega_{10}}\right| = \frac{z_1}{z_2}$$

## 12.9. Superposition of particular motions of rigid bodies. Relations for accelerations

In order to determine the acceleration of M with respect to the fixed Cartesian frame *Oxyz* it shall be considered the acceleration of M with respect to $O_1x_1y_1z_1$ as a *relative acceleration*, the acceleration of the point belonging to the $O_1x_1y_1z_1$ and coinciding with M as an *acceleration of transport* and it will also be considered the Coriolis acceleration. It follows that:

$$\begin{aligned}\bar{a}_r &= \bar{a}_{21} + \bar{\varepsilon}_{21} \times \bar{r}_2 + \bar{\omega}_{21} \times (\bar{\omega}_{21} \times \bar{r}_2)\\ \bar{a}_t &= \bar{a}_{10} + \bar{\varepsilon}_{10} \times r_1 + \bar{\omega}_{10} \times (\bar{\omega}_{10} \times \bar{r}_1)\\ \bar{a}_c &= 2\bar{\omega}_{10} \times (\bar{v}_{21} + \bar{\omega}_{21} \times \bar{r}_2)\end{aligned} \tag{12.31}$$

Consequently:

$$\begin{aligned}\bar{a}_M &= \bar{a}_{10} + \bar{a}_{21} + \bar{\varepsilon}_{10} \times \bar{r}_1 + \bar{\varepsilon}_{21} \times \bar{r}_2 + \bar{\omega}_{10} \times (\bar{\omega}_{10} \times \bar{r}_1) + \\ &+ \bar{\omega}_{21} \times (\bar{\omega}_{21} \times \bar{r}_2) + 2\bar{\omega}_{10} \times (\bar{v}_{21} + \bar{\omega}_{21} \times \bar{r}_2).\end{aligned} \tag{12.32}$$

The last formula can be generalized for *n* moving Cartesian frames $O_1x_1y_1z_1, \ldots\ldots O_nx_ny_nz_n$. It is obtained:

$$\begin{aligned}\bar{a}_M &= \sum_{i=1}^{n}\left[\bar{a}_{i,i-1} + \bar{\varepsilon}_{i,i-1} \times \bar{r}_i + \bar{\omega}_{i,i-1} \times \left(\bar{\omega}_{i,i-1} \times \bar{r}_i\right)\right]\\ &\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1}\left[\bar{\omega}_{j,j-1} \times \left(\bar{v}_{i,i-1} + \bar{\omega}_{i,i-1} \times \bar{r}_i\right)\right].\end{aligned} \tag{12.33}$$

Another point N belonging to the rigid body is now considered. The following relation holds:

$$\bar{a}_N = \bar{a}_M + \bar{\varepsilon}_{no} \times \overline{MN} + \bar{\omega}_{no} \times (\bar{\omega}_{no} \times \overline{MN}). \tag{12.34}$$

In this formula, $\bar{a}_M$ has the expression (12.33) and $\bar{\omega}_{no}$ has the expression (12.23). In order to determine $\bar{\varepsilon}_{n0}$, it is necessary to differentiate the relation (12.23). A certain vector $\bar{\omega}_{i,i-1}$ is defined by its projections on the axes of the moving Cartesian frame $O_ix_iy_iz_i$. By virtue of (12.4) it follows that:

$$\begin{aligned}\frac{d\bar{\omega}_{i,i-1}}{dt} &= \frac{\partial \bar{\omega}_{i,i-1}}{\partial t} + \bar{\omega}_{i0} \times \bar{\omega}_{i,i-1} = \bar{\varepsilon}_{i,i-1} + \left(\sum_{j=1}^{i}\bar{\omega}_{j,j-1}\right) \times \bar{\omega}_{i,i-1}\\ &= \bar{\varepsilon}_{i,i-1} + \left(\sum_{j=1}^{i-1}\bar{\omega}_{j,j-1}\right) \times \bar{\omega}_{i,i-1}\end{aligned} \tag{12.35}$$

Consequently:

$$\bar{\varepsilon}_{no} = \frac{d\bar{\omega}_{no}}{dt} = \sum_{i=1}^{n}\frac{d\bar{\omega}_{i,i-1}}{dt} = \sum_{i=1}^{n}\bar{\varepsilon}_{i,i-1} + \sum_{i=1}^{n}\sum_{j=1}^{i-1}\bar{\omega}_{j,j-1} \times \bar{\omega}_{i,i-1}. \tag{12.36}$$

It follows that the absolute angular acceleration $\bar{\varepsilon}_{no}$ is not equal only to the sum of relative angular accelerations $\bar{\varepsilon}_{i,i-1}$. A complementary angular acceleration:

$$\bar{\varepsilon}_c = \sum_{i=1}^{n}\sum_{j=1}^{i-1}\bar{\omega}_{j,j-1}\times\bar{\omega}_{i,i-1}, \tag{12.37}$$

must be added to this sum.

Example:

The disc of radius *r* rolls on a circle of the radius R (see figure) with constant angular velocity $\omega_{21}$. Determine the velocity and the acceleration of an arbitrary point M of the disc.

Denote by *Oxyz* the fixed frame (attached to the Earth), by $O_1x_1y_1z_1$ a moving frame that rotates about a vertical axis passing through the center of the circle of the radius R and by $O_2x_2y_2z_2$ a moving frame attached to the disc. Suppose that the axes of these three frames are coinciding at the considered time. The coordinates of a certain point M of the disc are *x, y, R*.

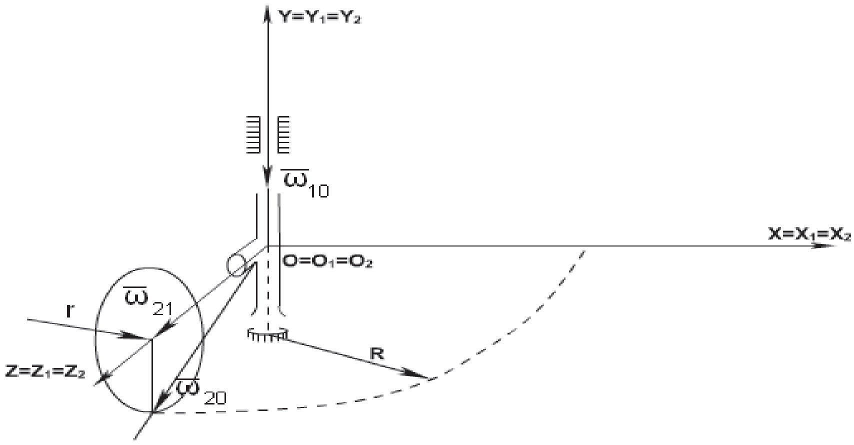



**Fig. 12.7 A disc rolling around a vertical axis**

From the given data and the orientations of the given vectors it follows:

$$\bar{\omega}_{10} = -\frac{r}{R}\omega\bar{j};\quad \bar{\omega}_{21} = \omega\bar{k};\quad \bar{\varepsilon}_{10} = 0;\quad \bar{\varepsilon}_{21} = 0;\ \bar{v}_{10} = 0;\ \bar{v}_{21} = 0;\ \bar{a}_{10} = 0;\ \bar{a}_{21} = 0;$$

$$\bar{r} = x\bar{i} + y\bar{j} + R\bar{k}$$

Applying the formulas (12.28) and (12.36), it can be obtained:

$$\bar{\omega}_{20} = \bar{\omega}_{10} + \bar{\omega}_{21} = -\frac{r}{R}\omega\bar{j} + \omega\bar{k};\quad \bar{\varepsilon}_{20} = \bar{\varepsilon}_{10} + \bar{\varepsilon}_{21} + \bar{\omega}_{10}\times\bar{\omega}_{21} = -\frac{r\omega^2}{R}\bar{i}$$

The velocity and the acceleration of a point M become:

$$\bar{v} = \bar{v}_0 + \bar{\omega}_{20}\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & -\frac{r}{R}\omega & \omega \\ x & y & R \end{vmatrix} = -(y+r)\omega\bar{i} + x\omega\bar{j} + \frac{r}{R}x\omega\bar{k},$$

$$\bar{a} = \bar{a}_O + \bar{\varepsilon}_{20} \times \bar{r} + \bar{\omega}_{20} \times (\bar{\omega}_{20} \times \bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\frac{r\omega^2}{R} & 0 & 0 \\ x & y & R \end{vmatrix} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & -\frac{r}{R}\omega & \omega \\ -(y+r)\omega & x\omega & \frac{r}{R}x\omega \end{vmatrix}$$

$$= -\left(1 + \frac{r^2}{R^2}\right) x\omega^2 \bar{i} - y\omega^2 \bar{j} - (r + 2y)\frac{r\omega^2}{R}\bar{k}$$

If $M$ is for example successively $A(0,-r,R)$, $B(0,r,R)$, $C(-r,0,R)$, $D(r,0,R)$, $E(0,0,R)$, the velocities and accelerations are:

$$\bar{v}_A = 0; \quad \bar{v}_B = -2r\omega\bar{i}; \quad \bar{v}_C = -r\omega\bar{i} - r\omega\bar{j} - \frac{r^2}{R}\omega\bar{k};$$

$$\bar{v}_D = -r\omega\bar{i} + r\omega\bar{j} + \frac{r^2}{R}\omega\bar{k}; \quad \bar{v}_E = -r\omega\bar{i}$$

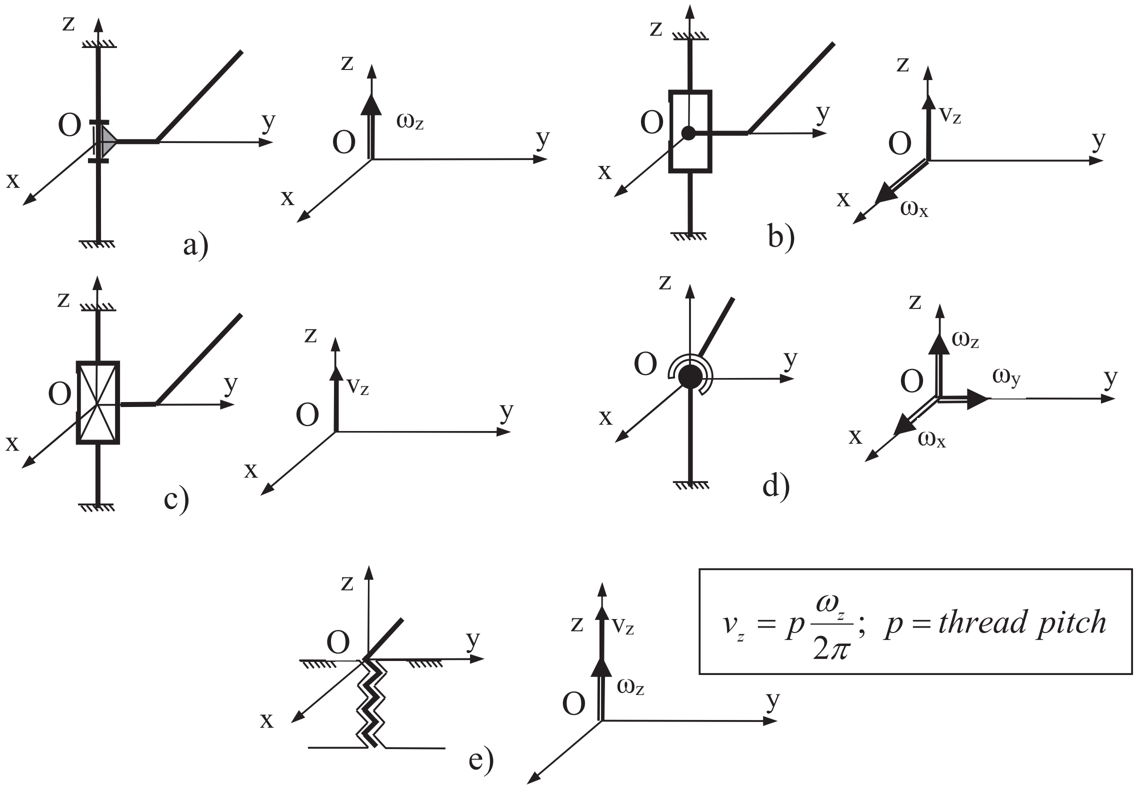
$$\bar{a}_A = r\omega^2\bar{j} + \frac{r^2\omega^2}{R}\bar{k}; \quad \bar{a}_B = -r\omega^2\bar{j} - 3\frac{r^2\omega^2}{R}\bar{k}; \quad \bar{a}_C = \left(1 + \frac{r^2}{R^2}\right) r\omega^2\bar{i} - \frac{r^2\omega^2}{R}\bar{k};$$

$$\bar{a}_D = -\left(1 + \frac{r^2}{R^2}\right) r\omega^2\bar{i} - \frac{r^2\omega^2}{R}\bar{k}; \quad \bar{a}_E = -\frac{r^2\omega^2}{R}\bar{k}$$

### *12.10. Fundamentals of mechanisms kinematic analysis*

A mechanism is a system of bodies designed to convert motions of one or several bodies into constrained motions of other bodies. A solid body or fluid component of a mechanism is called a mechanism element. A mechanism element carrying kinematical paring elements is called a link. Contacting elements of links permitting their constrained relative motion are called a kinematical pair. Examples of kinematical pairs are shown in Fig. 12.8 with their degree of freedom.

The kinematical pairs have simple independent motions such as translations or rotations, required to generate the relative motion of pairing elements. Examples are: the revolute pair (Fig. 12.8 a) which allows the rotation of one link relative to the another, the cylindrical pair (Fig. 12.8 b) which allows a rotation about a particular axis together with an independent translation in the direction of this axis, the prismatic pair (Fig. 12.8 c) which allows only a rectilinear translation of one link relative to another, the spherical pair (Fig. 12.8 d) which allows independent rotations about three concurrent axes and the helical (screw) pair (Fig. 12.8e) which allows a screw motion of the two links.




$$v_z = p\frac{\omega_z}{2\pi};\ p = thread\ pitch$$

**Fig. 12.8 Examples of mechanisms**

A number of rigid links connected by kinematical pairs form a kinematical chain. A kinematical chain may be open (Fig. 12.9 a ) or closed (Fig. 12.9 b ).
A mechanism is a kinematical chain with a defined motion. The mechanism element assumed to be stationary or to be either a support or foundation is called **frame**. The mechanism element that transfers mechanical energy at least to one other mechanism element directly connected to it is called a **link**.

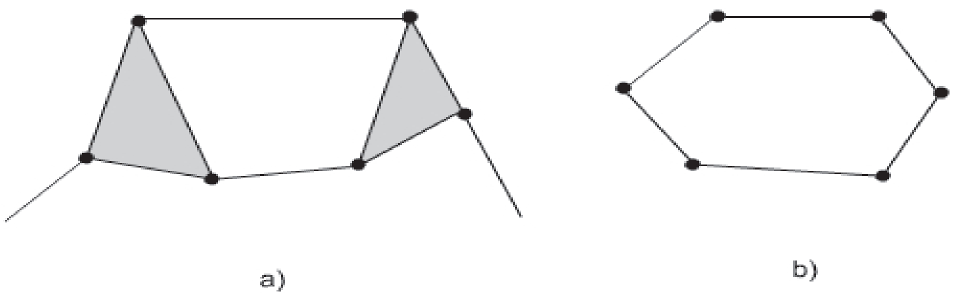



**Fig. 12.9 Open and closed kinematical chains**

A mechanism may have one or more driving links. Two mechanisms are shown in Fig. 12.10: the first has one driving link while the second has three driving links.

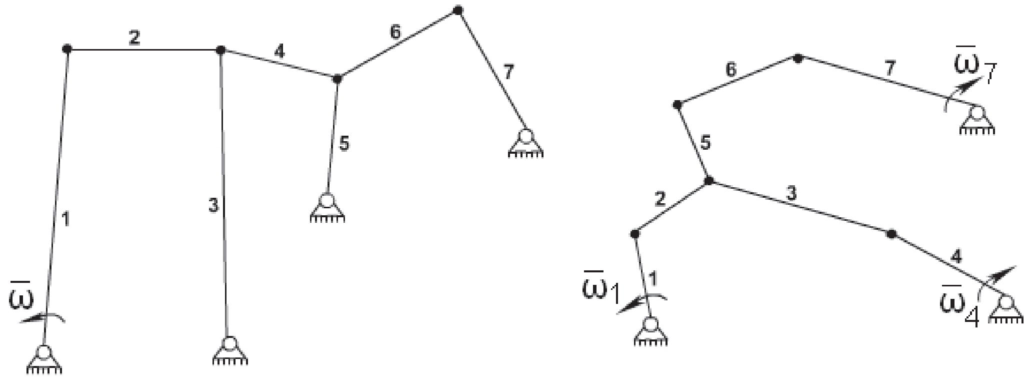



**Fig. 12.10 Mechanisms with one or several driving links**

### 12.10.1. Position analysis of a mechanism. Condition equations

It is considered a loop of a plane mechanism with *n* links and *n* rotation pairs and a polygonal contour with *n* line segments having their extremities in rotation pairs each line segment being attached to a link (Fig. 12.11).

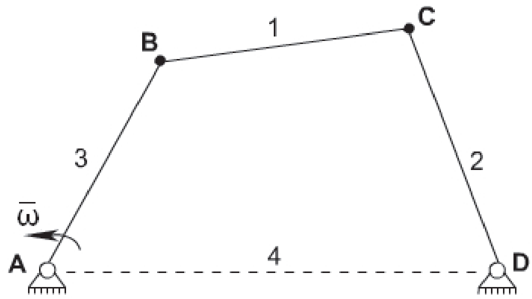



**Fig. 12.11 A plane mechanism with four segments**

The line segments can be considered associated to geometric vectors $\bar{l}_i$. It will be denoted by $x_i$ and $y_i$ the projections of the vector on the axes of a fixed Cartesian frame Oxy. Also by $l_i$ will be denoted the constant modulus of this vector. The following condition equations are evident:

$$\sum x_i = 0; \quad \sum y_i = 0; \quad x_i^2 + y_i^2 = l_i^2; \quad i = 1...n \qquad (12.38)$$

If the plane mechanism has more loops, the condition equations (12.38) can be written for each independent loop.

If the plane mechanism has a prismatic pair between the links $i$ and $i+1$ the angle $\alpha_{i,i+1}$ is constant and a new condition equation can be written

$$x_i x_{i+1} + y_i y_{i+1} = l_i l_{i+1} \cos\alpha_{i,i+1} \qquad (12.39)$$

Note that in this case, $l_i$ (or $l_{i+1}$) is not constant but it becomes a new unknown. Therefore a prismatic pair introduces a new unknown and a new condition equation.

In general a mechanism has a frame, a driving link and driven links. Note that only the projections $x_i$ and $y_i$ of the vectors $\overline{l_i}$ corresponding to driven links are unknown. Obviously, the projections of the vector corresponding to the frame are known. The projections $x_m$ and $y_m$ of driving link at time $t$ may be determined.
If $\theta$ is the angle between the Ox axis and the vector corresponding to the driving link at time $t$ and $\theta + \Delta\theta$ is the same angle at time $t + \Delta t$ it follows that:

$$x_m(t) = l_m\cos(\theta); y_m(t) = l_m\sin(\theta);$$

$$x_m(t+\Delta t) = l_m\cos(\theta+\Delta\theta) = l_m\cos\theta\cos\Delta\theta - l_m\sin\theta\sin\Delta\theta = x_m(t)\cos\Delta\theta - y_m(t)\sin\Delta\theta \quad (12.40)$$

$$y_m(t+\Delta t) = l_m\sin(\theta+\Delta\theta) = l_m\sin\theta\cos\Delta\theta + l_m\cos\theta\sin\Delta\theta = x_m(t)\sin\Delta\theta + y_m(t)\cos\Delta\theta$$

These relations may be written in a matrix form:

$$\begin{pmatrix} x_m(t+\Delta t) \\ y_m(t+\Delta t) \end{pmatrix} = \begin{bmatrix} \cos\Delta\theta & -\sin\Delta\theta \\ \sin\Delta\theta & \cos\Delta\theta \end{bmatrix} \begin{pmatrix} x_m(t) \\ y_m(t) \end{pmatrix} \quad (12.41)$$

If $x_m(0)$ and $y_m(0)$ are known and for example if $\Delta\theta = 10^o$ it is possible (by using (12.41)) to calculate the projections $x_m$ and $y_m$ for 36 positions of the driving link.
If the mechanism is a three dimensional spatial one the condition equations are more complicated. Considered here is a loop of a three dimensional mechanism with $n$ links and $n$ rotation pairs.

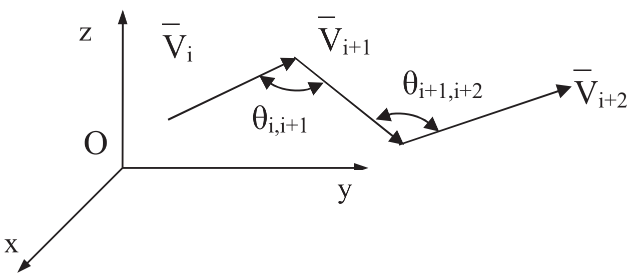



**Fig. 12.12 A three dimensional mechanism link**

A polygonal contour with $2n$ line segments: some line segments are placed on the rotation pair axes; other line segments are intersecting these axes. Three successive line segments of the polygonal contour belonging to the same link are orientated by the vectors $\overline{v}_i, \overline{v}_{i+1}$ and $\overline{v}_{i+2}$. Denote by $\theta_{i,i+1}$ and $\theta_{i,i+2}$ the angles of vectors $\overline{v}_i, \overline{v}_{i+1}$ and of the vectors $\overline{v}_i, \overline{v}_{i+2}$.

Obviously these angles (and also the angle $\theta_{i+1,i+2}$ of vectors $\overline{v}_{i+1}, \overline{v}_{i+2}$) are constant. Denote by $x_i$, $y_i$, $z_i$ the projections of the vector $\overline{v}_i$ on the axes of a fixed Cartesian frame Oxyz and by $l_i$ the constant modulus of $\overline{v}_i$ (the length of the line segment).
The following condition equations are evident:

$$
\begin{aligned}
&\sum x_i = 0; \qquad \sum y_i = 0; \qquad \sum z_i = 0; \\
&x^2 + y^2 + z^2 = l^2 \qquad (i = 1.....2n) \\
&x_i x_{i+1} + y_i y_{i+1} + z_i z_{i+1} = l_i l_{i+1} \cos\theta_{i,i+1} \qquad (i = 1.......2n) \\
&x_i x_{i+2} + y_i y_{i+2} + z_i z_{i+2} = l_i l_{i+2} \cos\theta_{i,i+2} \qquad (i = 1.....2n)
\end{aligned}
\tag{12.42}
$$

with the convention for indexes: $(2n+1) \equiv 1,\ (2n+2) \equiv 2$

If a pair is a cylindrical or prismatic one (but not a rotation pair) then the length of the corresponding line segment of the polygonal contour is no more constant and it becomes a new unknown. In the case of a prismatic pair a new condition equation may be written, because the angle between the two line segments is constant. If a pair is a spherical one, then 5 line segments of the polygonal contour may be considered to be equal to zero.

Note that only the projections $X_i$, $Y_i$, $Z_i$ of the vectors attached to the driven links are unknown. Obviously, the projections of the vectors attached to the frame are known. The projections $X_m$, $Y_m$, $Z_m$ of a vector attached to the driving link at the time $t$ may be determined. If $\theta$ is the angle of rotation of driving link (measured from an initial position for which $\theta=\theta_0$) at time $t$ and $\theta+\Delta\theta$ is the same angle at time $t+\Delta t$ the relation between $X_m(t+\Delta t)$, $Y_m(t+\Delta t)$, $Z_m(t+\Delta t)$ and $X_m(t)$, $Y_m(t)$, $Z_m(t)$ may be written in the matrix form :

$$
\begin{bmatrix} X_m \\ Y_m \\ Z_m \end{bmatrix} =
\begin{bmatrix}
\cos\theta + \alpha^2(1-\cos\theta) & \alpha\beta(1-\cos\theta) - \gamma\sin\theta & \alpha\gamma(1-\cos\theta) + \beta\sin\theta \\
\beta\alpha(1-\cos\theta) + \gamma\sin\theta & \cos\theta + \beta^2(1-\cos\theta) & \beta\gamma(1-\cos\theta) - \alpha\sin\theta \\
\gamma\alpha(1-\cos\theta) - \beta\sin\theta & \gamma\beta(1-\cos\theta) + \alpha\sin\theta & \cos\theta + \gamma^2(1-\cos\theta)
\end{bmatrix}
\begin{bmatrix} X_m(0) \\ Y_m(0) \\ Z_m(0) \end{bmatrix}
\tag{12.43}
$$

where $\alpha$, $\beta$ and $\gamma$ are direction cosines of the axis of rotation.
If $X_m(0)$, $Y_m(0)$ and $Z_m(0)$ are known and for example at $\theta=10°$, it is possible (using the formula (12.43)) to calculate the projections $X_m$, $Y_m$ and $Z_m$ for 36 positions of the driving link. The condition equations form a nonlinear system. If the unknowns are denoted by $x_1,...,x_N$ this system can be written in the general form:

$$
\begin{aligned}
f_1(x_1, ..., x_N, t) &= 0 \\
f_N(x_1, ..., x_N, t) &= 0
\end{aligned}
\tag{12.44}
$$

or simpler in matrix form :

$$\{ f(x,t) \} = \{0\} \tag{12.45}$$

where

$$\{ f(x,t) \}^T = [f_i, ..., f_N] \tag{12.46}$$

Given are the initial positions of all the links of the mechanism

$$\{ f(x_0, t) \} = \{0\} \tag{12.47}$$

It is assumed that $\theta$ is very small. Then $X_m(\Delta t) - X_m(0)$, $Y_m(\Delta t) - Y_m(0)$, $Z_m(\Delta t) - Z_m(0)$ are also small. By replacing in (12.45) the projection $X_m(0)$, $Y_m(0)$, $Z_m(0)$ by $X_m(\Delta t)$ $Y_m(\Delta t)$ $Z_m(\Delta t)$, then:

$$\{f(x,\Delta t)\} \neq \{0\} \tag{12.48}$$

but it is also very small. The Newton – Ralphson method can be used. The following recurrence formula is obtained:

$$\{x\}_n = \{x\}_{n-1} - (J)_{n-1}^{-1}\{f\}_{n-1} \tag{12.49}$$

where (J) is the matrix of the Jacobian of { f } i.e. :

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots\cdots\cdots \frac{\partial f_1}{\partial x_N} \\ \cdots \\ \frac{\partial f_N}{\partial x_1} \cdots\cdots\cdots \frac{\partial f_N}{\partial x_N} \end{bmatrix}. \tag{12.50}$$

Suppose the solutions $\{f(x,\Delta t)\} \neq \{0\}$ is known. By replacing in this equation $X_m(\Delta t)$, $Y_m(\Delta t)$, $Z_m(\Delta t)$ by $X_m(2\Delta t)$, $Y_m(2\Delta t)$, $Z_m(2\Delta t)$ then $\{f(x,2\Delta t)\} \neq \{0\}$, but it is very small. The Newton Ralphson method can be used once again. If $\Delta\theta$ is small e.g. $10°$, it is possible to obtain 36 positions of the given mechanism.

### 12.10.2. Kinematic analysis of a mechanism

In order to obtain the velocities, is necessary the derivative of the matrix equation with respect to the time t. The solution is:

$$(J)\{\dot{x}\} + \left\{\frac{\partial f}{\partial t}\right\} = \{0\} \tag{12.51}$$

It follows the expressions of the velocities:

$$\{\dot{x}\} = -(J)^{-1}\left\{\frac{\partial f}{\partial t}\right\} \tag{12.52}$$

If $X_m(t)$, $Y_m(t)$, $Z_m(t)$ are the projections of a vector attached to the driving link, then $\dot{X}_m, \dot{Y}_m, \dot{Z}_m$ (the elements of $\left\{\frac{\partial f}{\partial t}\right\}$ differing from zero) have the expressions:

$$\begin{aligned} \dot{X}_m &= (\beta Z_m - \gamma Y_m)\omega \\ \dot{Y}_m &= (\gamma X_m - \alpha Z_m)\omega \\ \dot{Z}_m &= (\alpha Y_m - \beta X_m)\omega \end{aligned} \tag{12.53}$$

For a plane mechanism these expressions become:

$$\dot{X}_m = -Y_m\omega$$
$$\dot{Y}_m = X_m\omega \tag{12.54}$$

In order to obtain the accelerations the derivative of the matrix equation (12.51) with respect to the time t must be obtained. The result is:

$$(J)\langle\ddot{x}\rangle + (\dot{J})\{\dot{x}\} + \left\{\frac{\partial^2 f}{\partial t^2}\right\} = \{0\} \tag{12.55}$$

It follows the expressions of accelerations:

$$\{\ddot{x}\} = -(J)^{-1}\left\{\frac{\partial^2 f}{\partial t^2}\right\} + (J)^{-1}(\dot{J})(J)^{-1}\left\{\frac{\partial f}{\partial t}\right\} \tag{12.56}$$

The components of $\left\{\frac{\partial^2 f}{\partial t^2}\right\}$ differing from zero, $\ddot{X}_m, \ddot{Y}_m$ and $\ddot{Z}_m$ have the expressions:

$$\ddot{X}_m = (\beta Z_m - \gamma Y_m)\varepsilon - \left[(\beta^2+\gamma^2)X_m - \alpha\beta Y_m - \alpha\gamma Z_m\right]\omega^2$$
$$\ddot{Y}_m = (\gamma X_m - \alpha Z_m)\varepsilon - \left[-\beta\alpha X_m + (\gamma^2+\alpha^2)Y_m - \beta\gamma Z_m\right]\omega^2 \tag{12.57}$$
$$\ddot{Z}_m = (\alpha Y_m - \beta X_m)\varepsilon - \left[-\gamma\alpha X_m - \gamma\beta Y_m + (\alpha^2+\beta^2)Z_m\right]\omega^2$$

For a plane mechanism, these expressions become:

$$\ddot{X}_m = Y_m\varepsilon - X_m\omega^2$$
$$\ddot{Y}_m = X_m\varepsilon - Y_m\omega^2 \tag{12.58}$$

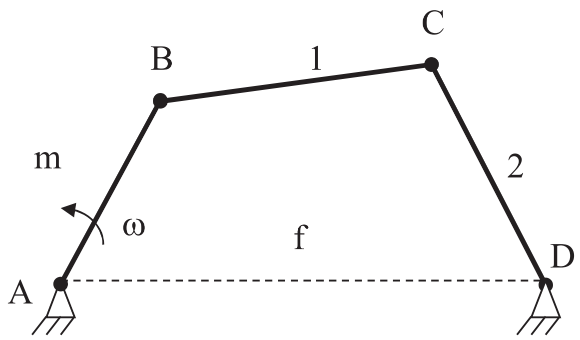



**Fig. 12.13 A quadrilateral mechanism**

Example 1

Write the condition equations and the expression of $(J)$ and $(\dot{J})$ for the plane mechanism shown on Fig. 12.13. All lengths are given.

The condition equations are:

$$X_m + X_1 + X_2 + X_f = 0$$
$$Y_m + Y_1 + Y_2 + Y_f = 0$$
$$X_1^2 + Y_1^2 - l_1^2 = 0$$
$$X_2^2 + Y_2^2 - l_2^2 = 0$$

The expressions of $(J)$ and $(\dot{J})$ are:

$$\left(J\right)=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2X_1 & 0 & 2Y_1 & 1 \\ 0 & 2X_2 & 0 & 2Y_2 \end{bmatrix}; \quad \left(\dot{J}\right)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\dot{X}_1 & 0 & 2\dot{Y}_1 & 0 \\ 0 & 2\dot{X}_2 & 0 & 2\dot{Y}_2 \end{bmatrix}$$

Example 2

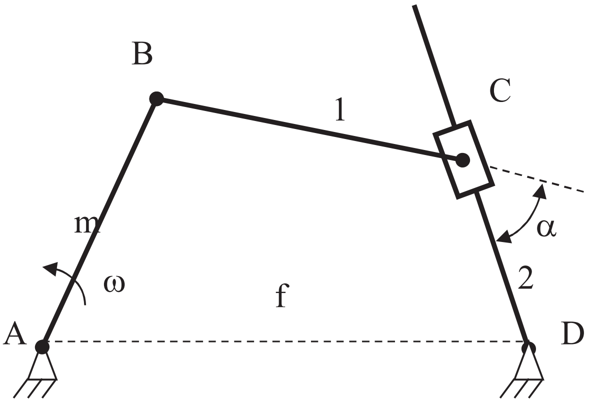



Fig. 12.14 A quadrilateral mechanism with an internal slider

Write the conditions equations and the expressions of $\left(J\right)$ and $\left(\dot{J}\right)$ for the plane mechanism shown on Fig. 12.14.

The condition equations are:

$$X_m + X_1 + X_2 + X_f = 0$$
$$Y_m + Y_1 + Y_2 + Y_f = 0$$
$$X_1^2 + Y_1^2 - l_1^2 = 0$$
$$X_2^2 + Y_2^2 - l_2^2 = 0$$
$$X_1 X_2 + Y_1 Y_2 - l_1 l_2 \cos\alpha = 0$$

The expressions of (J) and $\left(\dot{J}\right)$ are:

$$\left(J\right)=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 2X_1 & 0 & 2Y_1 & 1 & -2l_1 \\ 0 & 2X_2 & 0 & 2Y_2 & 0 \\ X_2 & X_1 & Y_2 & Y_1 & -l_2\cos\alpha \end{bmatrix}; \quad \left(\dot{J}\right)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2\dot{X}_1 & 0 & 2\dot{Y}_1 & 0 & -2\dot{l}_1 \\ 0 & 2\dot{X}_2 & 0 & 2\dot{Y}_2 & 0 \\ \dot{X}_2 & \dot{X}_1 & \dot{Y}_2 & \dot{Y}_1 & 0 \end{bmatrix}$$

# 13. DYNAMICS OF A MATERIAL POINT

## 13.1. Dynamics of a free material point

It is assumed a free material point and Newtonian force $\overline{F}$ acting on it, i.e. a force depending only on the position vector $\overline{r}$, or also on the velocity $\overline{v}$ and possibly depending explicitly on the time t. The fundamental law of dynamics can be written:

$$m\overline{a} = \overline{F}\left(\overline{r}, \overline{v}, t\right). \tag{13.1}$$

The projections on the axes of a fixed Cartesian frame of this vector equation are

$$\begin{aligned} m\ddot{x} &= X\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \\ m\ddot{y} &= Y\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \\ m\ddot{z} &= Z\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \end{aligned} . \tag{13.2}$$

These equations form a system of three differential equations of second order. The **general solutions** depend on six constants. Suppose that these solutions are

$$\begin{aligned} x &= f\left(t, C_1, C_2, C_3, C_4, C_5, C_6\right) \\ y &= g\left(t, C_1, C_2, C_3, C_4, C_5, C_6\right) \\ z &= h\left(t, C_1, C_2, C_3, C_4, C_5, C_6\right) \end{aligned} . \tag{13.3}$$

To determine these constants, it is necessary to know the **initial conditions** (for t=0). Suppose that these initial conditions are:

$$\begin{aligned} &x = x_0; \quad y = y_0; \quad z = z_0; \\ &\dot{x} = v_{x0}; \quad \dot{y} = v_{y0}; \quad \dot{z} = v_{z0} \end{aligned} \tag{13.4}$$

It follows that

$$\begin{aligned} f\left(0, C_1, C_2, C_3, C_4, C_5, C_6\right) &= x_0 \\ g\left(0, C_1, C_2, C_3, C_4, C_5, C_6\right) &= y_0 \\ h\left(0, C_1, C_2, C_3, C_4, C_5, C_6\right) &= z_0 \\ f'\left(0, C_1, C_2, C_3, C_4, C_5, C_6\right) &= v_{x0} \\ g'\left(0, C_1, C_2, C_3, C_4, C_5, C_6\right) &= v_{y0} \\ h'\left(0, C_1, C_2, C_3, C_4, C_5, C_6\right) &= v_{z0} \end{aligned} . \tag{13.5}$$

These six equations form a system of algebraic equations permitting in many cases of interest to express the six constants $C_1$,..., $C_6$ with respect to the given initial conditions. If these six constants are obtained by solving the system of (usually nonlinear) equations(13.5) and are afterwards injected into the general solution

(13.3), then is obtained the **particular solution**:

$$\begin{aligned} x &= x\left(t, x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0}\right) \\ y &= y\left(t, x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0}\right). \\ z &= z\left(t, x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0}\right) \end{aligned} \tag{13.6}$$

This particular solution represents parametric form of the equation of motion really taking place.

### *13.2. Motion of a heavy free material point neglecting air drag*

A fixed Cartesian frame Oxyz (Fig. 13.1) is considered. The origin O is in the initial position of the material point; the Oy axis vertical and the Ox axis placed such that the initial velocity $\overline{v}_0$ is situated in the vertical plane Oxy.

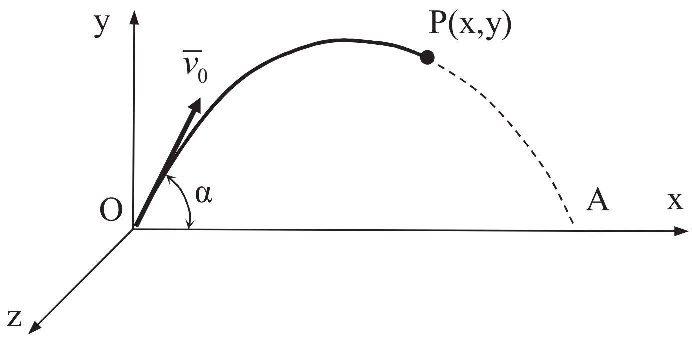



**Fig. 13.1 Free motion of a material point in gravitational field and in vacuum**

The fundamental equation of dynamics becomes:

$$m\overline{a} = m\overline{g}\,. \tag{13.7}$$

The projections of this equation on the axes of the Cartesian frame Oxyz (Fig. 13.1) are:

$$\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= -g\,. \\ \ddot{z} &= 0 \end{aligned} \tag{13.8}$$

By successive integration of these ordinary linear differential equations of second order, the following expressions for velocity and acceleration are obtained:

$$\begin{cases} \dot{x} = C_1 \\ \dot{y} = -gt + C_2\,; \\ \dot{z} = C_3 \end{cases} \qquad \begin{cases} x = C_1 t + C_4 \\ y = -g\dfrac{t^2}{2} + C_2 t + C_5 \\ z = C_3 t + C_6 \end{cases} \tag{13.9}$$

The initial conditions, according to above given information are:

$$\begin{cases} \dot{x} = v_0 \cos\alpha \\ \dot{y} = v_0 \sin\alpha \\ \dot{z} = 0 \end{cases}; \qquad \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad (13.10)$$

It follows that:

$$\begin{cases} C_1 = v_0 \cos\alpha \\ C_2 = v_0 \sin\alpha \\ C_3 = 0 \end{cases}; \qquad \begin{cases} C_4 = 0 \\ C_5 = 0 \\ C_6 = 0 \end{cases} \qquad (13.11)$$

The particular solution, representing the motion is:

$$\begin{cases} x = v_0 t \cos\alpha \\ y = -g\dfrac{t^2}{2} + v_0 t \sin\alpha \\ z = 0 \end{cases} \qquad (13.12)$$

It follows that the path of the material point is situated into the Oxyz plane. Eliminating the time $t$ in the parametric equations(13.12), the equation of the path in the form:

$$y = -\frac{g}{2v_0^2 \cos^2\alpha} x^2 + x \tan\alpha \qquad (13.13)$$

Therefore the path is a second degree parabola.

Remarks:

a) The parabola intersects the Ox axis in two points O, the starting point and another point A (Fig. 13.1). If the material point is a projectile, the length of the segment OA is called **range** of the projectile. The projectile strikes the ground when y = 0, so the two points verify the condition: $0 = -\frac{g}{2v_0^2 \cos^2\alpha} x^2 + x \tan\alpha$, from which

$$x_O = 0$$
$$x_A = \frac{2v_0^2 \sin\alpha \cos\alpha}{g} = \frac{v_0^2}{g}\sin 2\alpha \qquad (13.14)$$

Note that the range of the projectile reaches its maximum for $\alpha = 45^\circ$, which is the result of $\frac{dx_A}{d\alpha} = 2\frac{v_0^2}{g}\cos 2\alpha = 0$. It follows that $(x_A)_{\max} = \frac{v_0^2}{g}$.

b) The highest point attained by the projectile on the parabolic path is obtained by letting $\frac{dy}{dx} = -\frac{g}{v_0^2 \cos^2 \alpha} x + \tan \alpha = 0$.

It follows that:

$$x_{H\max} = \frac{v_0^2}{g} \sin \alpha \cos \alpha = \frac{v_0^2}{2g} \sin 2\alpha$$
$$y(x_{H\max}) = \frac{v_0^2}{2g} \sin^2 \alpha \qquad (13.15)$$

If the initial velocity has a vertical direction ($\alpha = 90^\circ$), the maximum height is:

$$y_{\max}(x_{H\max}) = \frac{v_0^2}{2g} \qquad (13.16)$$

c) A projectile can be launched with a velocity $\overline{v}_0$ in any direction. Which is the region in space that can be reached by the projectile?

The equation (13.13) can be written as

$$y + \frac{g}{2v_0^2}\left(1 + \tan^2 \alpha\right)x^2 - x \tan \alpha = \frac{gx^2}{2v_0^2} \tan^2 \alpha - x \tan \alpha + y + \frac{gx^2}{2v_0^2} = 0 \qquad (13.17)$$

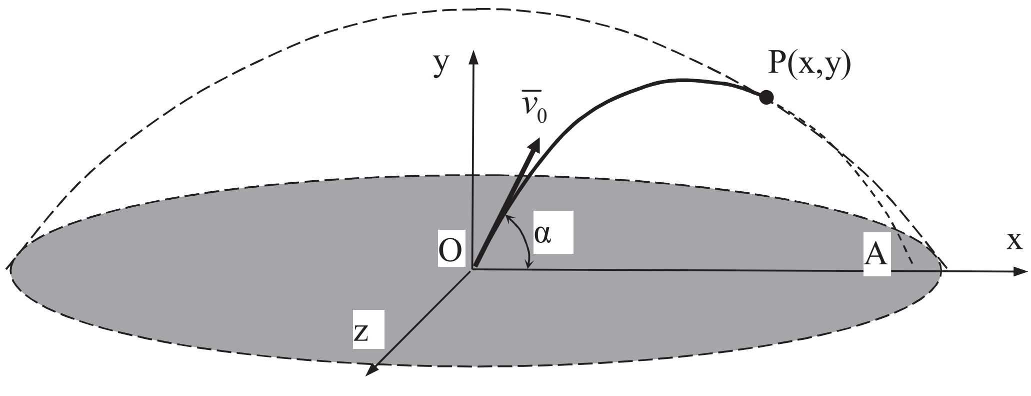



**Fig. 13.2 Geometrical locus of reachable positions for a launched projectile**

This equation can be considered as a second degree equation in (tan $\alpha$). Real solutions exist if and only if

$$x^2 - 4\frac{gx^2}{2v_0^2}\left(y + \frac{gx^2}{2v_0^2}\right) \geq 0, \qquad (13.18)$$

from which a condition relating y to x can be obtained:

$$y \le \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} \tag{13.19}$$

Therefore, the region in plane that can be reached by the projectile is the inside of a parabola (Fig. 13.2). The region in space shall be the inside of a paraboloid of revolution, possibly intersected by the ground surface.

## *13.3. Motion of a heavy free material point considering air drag*

Let $\overline{R}$ be the drag force applied on the moving particle (material point) in the air. This force is acting in the opposite direction of motion:

$$\overline{R}(\overline{v}) = -f(v)\overline{\tau}, \tag{13.20}$$

in which $\overline{\tau}$ is the unit vector tangent to the path of the particle and *f(v)* is a positive definite, monotonic increasing real function. The fundamental equation of dynamics can be written:

$$m\overline{a} = m\overline{g} - f(v)\overline{\tau}. \tag{13.21}$$

The projections of this equation on the axes of the moving frame of Serret - Frenet are:

$$\begin{cases} m\ddot{s} = m\dot{v} = -mg\sin\theta - f(v) \\ m\dfrac{v^2}{\rho} = mg\cos\theta \end{cases} \tag{13.22}$$

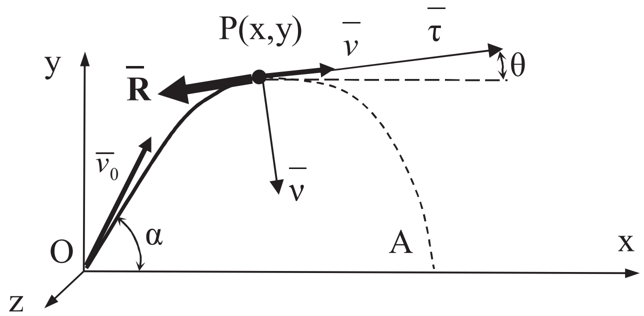



**Fig. 13.3 Free motion of a material point in gravitational field and considering air drag force**

These equations can be rewritten using the following expressions:

$$\begin{aligned} &\dot{v} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = \frac{d}{ds}\left(\frac{v^2}{2}\right); \\ &\sin\theta = \frac{dy}{ds};\ \cos\theta = \frac{dx}{ds};\ \rho = \left|\frac{ds}{d\theta}\right| = -\frac{ds}{d\theta} \end{aligned} \tag{13.23}$$

In the last expression, the minus sign has been chosen from physical consideration of decreasing slope (θ) for the falling particle, keeping positive the curvature radius ρ. The new form of (13.22) is

$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{v^2}{2}\right) = -g\dfrac{dy}{ds} - \dfrac{f(v)}{m} \\ -\dfrac{v^2 d\theta}{ds} = g\dfrac{dx}{ds} = \dfrac{g}{\tan\theta}\dfrac{dy}{ds} \end{cases}. \tag{13.24}$$

Properties of the motion:

a) Considering two positions of the particle along its path, of identical height $y_M = y_N$ (M on the ascending arc MP of the path and N on the descendent arc PN), it can be stated that at the same height, the velocity in the ascending motion is higher than in the descending motion: $|v_M| > |v_N|$ (Fig. 13.4).

Indeed, by integrating the first equation (13.24) on the arc MN, it follows that:

$$\frac{v_N^2}{2} - \frac{v_M^2}{2} = y_N - y_M - \frac{1}{m}\oint_{MPN} f(v(s))ds = -\frac{1}{m}\oint_{MPN} f(v(s))ds \tag{13.25}$$

Since $f(v)$ is a positive definite function, it follows that: $v_N^2 < v_M^2$ or $|v_N| < |v_M|$

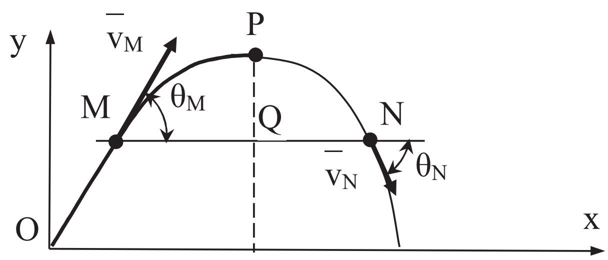



**Fig. 13.4 Velocity on the descending arc is less than on the ascending arc**

b) If $\theta_M$ and $\theta_N$ are the angles of the velocities $\bar{v}_M$ and respectively $\bar{v}_N$ with the Ox axis, then at identical height the angles with the horizontal direction are smaller in the ascending motion than in the descending one: $|\theta_M| < |\theta_N|$.

Indeed, by integrating the second equation (13.24) on the arc MN, it can be written:

$$\oint_{MN} \tan\theta d\theta = -g\oint_{MN} \frac{dy}{v^2}, \tag{13.26}$$

or after successive operations on the left and right terms:

$$\oint_{MN} \tan\theta d\theta = -\ln\left(\cos\theta_N\right) + \ln\left(\cos\theta_M\right) = \ln\frac{\cos\theta_M}{\cos\theta_N};$$
$$-g\oint_{MN}\frac{dy}{v^2} = -g\oint_{MP}\frac{dy}{v^2} - g\oint_{PN}\frac{dy}{v^2} = -g\oint_{MP}\frac{dy}{v^2} + g\oint_{NP}\frac{dy}{v^2} > 0. \tag{13.27}$$

The last inequality is based on property (a), velocities being higher on the MP arc than on the NP arc. The consequence is

$$\ln\frac{\cos\theta_M}{\cos\theta_N} > 0 \Rightarrow \cos\theta_M > \cos\theta_N \quad \Rightarrow \theta_M < \theta_N. \tag{13.28}$$

c) The horizontal projection of the path is larger on the ascending arc than on the descending arc: MQ > QN.
The integral of the second equation from (13.24) becomes:

$$MQ = \oint_{MP} dx = \oint_{MP}\frac{dy}{\tan\theta}; \quad QN = \oint_{PN} dx = \oint_{PN}\frac{dy}{\tan\theta} = \oint_{NP}\frac{dy}{\tan|\theta|}. \tag{13.29}$$

Since θ and implicitly tan θ are greater in modulus on the NP arc than on the e MP arc, it follows that MQ>QN.

d) The absolute value of the velocity reaches a minimum value on the descending branch of the path.

From the first equation (13.24) it follows that at the vertex P, the modulus of the velocity decreases, since:

$$\frac{d}{ds}\left(\frac{v^2}{2}\right) = -g\sin 0 - \frac{f(v)}{m} < 0. \tag{13.30}$$

For the descending branch of the path, the angles $\theta$ are negative, so that there could be an angle $\theta_m$ for which

$$g\sin\theta_m = \frac{f(v)}{m}, \tag{13.31}$$

which represents a condition of minimum for the absolute value of the velocity, due to the expression $\frac{d}{ds}\left(\frac{v^2}{2}\right) = 0$.

e) There is a limit velocity $v_L$, which is the solution of the equation: $mg = f(v_L)$.

The velocity decreases after passing the vertex P, until reaching the angle $\theta_m<0$ for which $f(v) = mg\sin(-\theta_m)$. After this point, the velocity might increase again as the angle θ should continue to drop towards the vertical direction: $-\pi/2 < \theta < \theta_m$. But exactly at this angle, the first equation (13.24) is:

$$\frac{d}{ds}\left(\frac{v^2}{2}\right) = -g\sin\theta_m - \frac{mg}{m} = -g\left(1+\sin\theta_m\right) \le 0 . \qquad (13.32)$$

It follows that after reaching this angle, the velocity continues to decrease, a fact which is incompatible with the previous minimum condition. It follows that the minimum velocity will be reached for a given position on the descending branch, but this position is at infinite distance from P.

f) The path has a vertical asymptote.
It will be assumed that the velocity on the descending branch increases up a finite limit value $v_L$. From the second equation (13.24) it follows:

$$-\int_0^{-\pi/2} v^2 d\theta = g\int_{x_P}^{x_\infty} dx . \qquad (13.33)$$

and integrating with the given limits:

$$x_\infty = x_P + \frac{v_P^2 - v_L^2}{g} , \qquad (13.34)$$

which is a finite distance, representing a vertical asymptote.

### 13.3.1. Case of drag force linearly dependent on velocity

A particle of mass m is launched with a velocity $\overline{v}_0$ in a vertical plane Oxy, making an angle α with the horizontal Ox axis. The drag force $\overline{R}$, is assumed to be dependent on velocity as:

$$\overline{R}\left(\overline{v}\right) = -cv\overline{\tau} = -c\overline{v} = -c\left(\dot{x}\overline{i} + \dot{y}\overline{j}\right) , \qquad (13.35)$$

in which *c* is a positive constant. The fundamental equation of dynamics can be written:

$$m\overline{a} = m\overline{g} - c\overline{v} . \qquad (13.36)$$

The projections of this equation on the axes of the fixed Oxy frame are:

$$\begin{cases} m\ddot{x} = -c\dot{x} \\ m\ddot{y} = -mg - c\dot{y} \end{cases} \qquad (13.37)$$

The characteristic equation of the homogeneous differential equation is $\lambda^2 + \beta\lambda = 0$, with two real solutions: $\lambda_1 = 0$ and $\lambda_2 = -\beta$, in which $\beta = \frac{c}{m}$. The general solution of the first differential equation is:

$$x\left(t\right) = C_1 + C_2 e^{-\beta t} . \qquad (13.38)$$

The second differential equation is non-homogeneous, but the homogeneous part is similar to the first equation. It means that a particular solution must be added to a similar solution, to get the general solution.

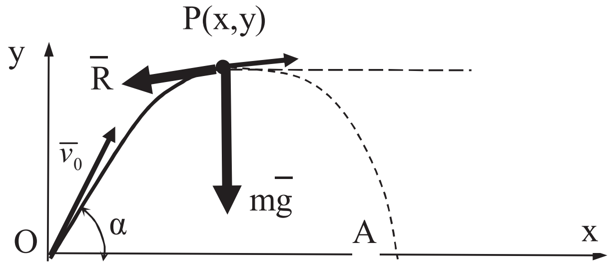



**Fig. 13.5 Free motion of a material point in gravitational field and considering air drag force**

One such particular solution is $y_p(t) = -\beta g t$, so that the general solution of the first differential equation is:

$$y(t) = C_3 + C_4 e^{-\beta t} - \frac{g}{\beta} t \,. \tag{13.39}$$

A system of algebraic equations in constants $C_1 \ldots C_4$ is obtained from the initial conditions:

$$\begin{aligned} & x(0) = C_1 + C_2 e^{-\beta 0} = 0 \\ & y(0) = C_3 + C_4 e^{-\beta 0} - \frac{g}{\beta} 0 = 0 \\ & \dot{x}(0) = -\beta C_2 e^{-\beta 0} = v_0 \cos\alpha \\ & \dot{y}(0) = -\beta C_4 e^{-\beta 0} - \frac{g}{\beta} = v_0 \sin\alpha \end{aligned} \tag{13.40}$$

This algebraic system of equations in $C_1$, $C_2$, $C_3$, $C_4$ can be solved, obtaining the solutions:

$$C_2 = -C_1 = -\frac{v_0}{\beta}\cos\alpha; \;\; C_4 = -C_3 = -\frac{1}{\beta}\left(\frac{g}{\beta} + v_0 \sin\alpha\right). \tag{13.41}$$

The solution to the particular given problem is:

$$\begin{aligned} & x(t) = \frac{v_0}{\beta}\cos\alpha\left(1 - e^{-\beta t}\right); \\ & y(t) = \frac{1}{\beta}\left(\frac{g}{\beta} + v_0 \sin\alpha\right)\left(1 - e^{-\beta t}\right) - \frac{g}{\beta} t \end{aligned} \tag{13.42}$$

The velocity of the particle is:

$$\dot{x}(t) = v_0 \cos\alpha e^{-\beta t}; \;\; \dot{y}(t) = \left(\frac{g}{\beta} + v_0 \sin\alpha\right) e^{-\beta t} - \frac{g}{\beta} \,. \tag{13.43}$$

Properties of the motion:

a) There is a limit velocity $v_L$, which is the solution of the equation: $mg = f(v_L)$.
Indeed:

$$\lim_{t\to\infty} \dot{x}(t) = 0; \quad \lim_{t\to\infty} \dot{y}(t) = -\frac{g}{\beta}, \tag{13.44}$$

so that $v_L = \sqrt{0^2 + \left(-\frac{g}{\beta}\right)^2} = \frac{g}{\beta}$, which is the solution of the equation $mg = cv_L$.

b) The path has a vertical asymptote.
The coordinate $x$ for theoretical infinite time is:

$$\lim_{t\to\infty} x(t) = \frac{v_0}{\beta}\cos\alpha, \tag{13.45}$$

which is a finite distance, representing thus a vertical asymptote. All the other properties proven in the general case can be verified in this particular case.

### *13.4. Motion of a free material point submitted to a central force*

The fundamental equation of dynamics becomes:

$$m\overline{a} = F\overline{\rho}, \tag{13.46}$$

in which F is the projection of a **central force**, meaning a force whose support line passes through a fixed point O at any moment in time and $\overline{\rho} = \frac{\overline{r}}{|\overline{r}|}$ is the corresponding unit vector (Fig. 13.6). If F>0, the central force is called **repulsive**, and **attractive** if F<0.

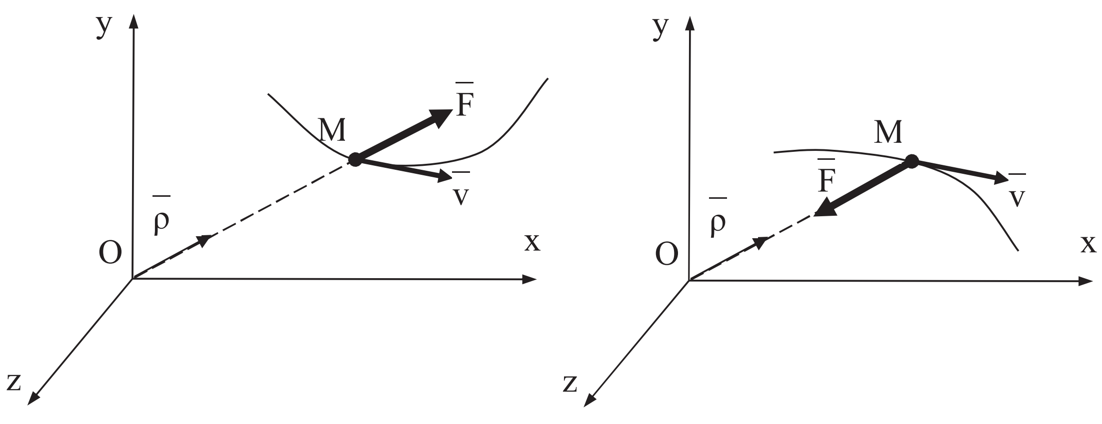



**Fig. 13.6 Motion of a particle in a field of central forces**

If $\overline{OM} = \overline{r}$ is the position vector of the material point M, it follows that: $\overline{F} \parallel \overline{r}$. Consequently the vector product $\overline{r} \times \overline{F} = 0$. The consequence is

$$\overline{r} \times \overline{a} = \overline{r} \times \frac{d\overline{v}}{dt} = d\left(\overline{r} \times \overline{v}\right) - \frac{d\overline{r}}{dt} \times \overline{v} = d\left(\overline{r} \times \overline{v}\right) - \overline{v} \times \overline{v} = d\left(\overline{r} \times \overline{v}\right) = \overline{0} \quad (13.47)$$

It follows that:

$$\overline{r} \times \overline{v} = \overline{C} \quad (13.48)$$

in which $\overline{C}$ is a constant vector (constants modulus and direction). The scalar product of this expression and the position vector is a null vector:

$$\overline{C} \cdot \overline{r} = \left(\overline{r} \times \overline{v}\right) \cdot \overline{r} = \left(\overline{r} \times \overline{r}\right) \cdot \overline{v} = 0 . \quad (13.49)$$

Assuming $C_1$, $C_2$, $C_3$ to be the Cartesian projections of $\overline{C}$ and x, y, z those of $\overline{r}$, the last relation can be rewritten:

$$C_1 x + C_2 y + C_3 z = 0 . \quad (13.50)$$

This last expression represents the equation of a fixed plane, passing through the origin O. The consequence is that the particle submitted to central forces has a planar path and the plane including the path is fixed.

The motion can be better studied in a system of polar coordinates included in the plane of the motion.

$$\begin{cases} m\left(\ddot{r} - r\dot{\theta}^2\right) = F \\ m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0 \end{cases} . \quad (13.51)$$

The second equation can be written:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 . \quad (13.52)$$

It follows that:

$$r^2\dot{\theta} = C . \quad (13.53)$$

It has been proven that $r^2\dot{\theta} = 2\Omega$ (§10.9) where Ω is the area velocity. It can be stated that during the motion of a material point submitted to a central force, the area velocity (or rate of sweeping out area) is a constant.

The relations (13.50) and (13.53) represent two general properties of the motion of a particle acted only by a central force, the first two laws of Kepler (1609,1619).
An important class of problems requires the path of the particle rather than its time dependency. In such cases time can be replaced in the differential equations, leaving two parameters: polar radius (r) and position angle (θ).

Using the changes of variable:

$$\dot{r}=\frac{dr}{dt}=\frac{dr}{d\theta}\frac{d\theta}{dt}=\frac{C}{r^2}\frac{dr}{d\theta}=-C\frac{d}{d\theta}\left(\frac{1}{r}\right)$$
$$\ddot{r}=\frac{d^2r}{dt^2}=\frac{d}{d\theta}\left[-C\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]\frac{d\theta}{dt}=-\frac{C^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right), \tag{13.54}$$

the first equation (13.51) becomes:

$$-\frac{C^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)-r\frac{C^2}{r^4}=\frac{F}{m}, \tag{13.55}$$

or after rearranging terms:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{Fr^2}{mC^2}. \tag{13.56}$$

This is known as Binet's equation.

**Example**

Determine the path of a free material point submitted to an universal attractive Newtonian force $\bar{F}=-f\frac{mM}{r^2}\frac{\bar{r}}{|\bar{r}|}$, where *f* is the gravitational constant ($6.67\cdot10^{-11}$ $m^3kg\cdot s^{-2}$), *M* is the mass of the point O and assumed to be fixed, *m* the mass of the material point and *r* the distance between the two points. Initial conditions are: initial radius $r_0$, initial velocity $v_0$ making an angle α with the initial radius direction.

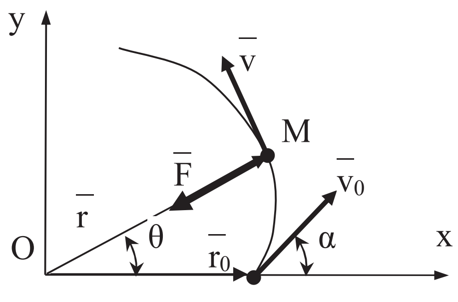



**Fig. 13.7 Central force motion of a particle in gravitational attraction field**

The Binet’s equation in this case is

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{fM}{C^2}. \tag{13.57}$$

The solution to equation(13.57), from elementary differential equations, consists of a homogeneous (or complementary) part plus a particular part.

The associated characteristic equation $\lambda^2+1=0$ has imaginary solutions. The homogeneous part is a sum of harmonic functions and the particular part has the form of the right side if the equation:

$$\frac{1}{r}=A\cos\theta+B\sin\theta+\frac{fM}{C^2}. \tag{13.58}$$

The two constants are obtained from the initial conditions (at t=0, r=$r_0$ and θ=0). The velocity in polar coordinates has components: $\dot{r}$ and $r\dot{\theta}$, requiring

$$-\frac{1}{r^2}\dot{r}=\left(-A\sin\theta+B\cos\theta\right)\dot{\theta}=\left(-A\sin\theta+B\cos\theta\right)\frac{C}{r^2}. \tag{13.59}$$

The two initial conditions are written as:

$$\begin{aligned}&\frac{1}{r_0}=A+\frac{fM}{C^2}\\&B\cdot C=-\dot{r}=-\left(v_0\cos\alpha\right)\end{aligned}. \tag{13.60}$$

Moreover the motion constant $C=(r^2\dot{\theta})\Big|_{t=0}=r_0^2\frac{v_0\sin\alpha}{r_0}=r_0v_0\sin\alpha$, so that the two constants become

$$\begin{aligned}&A=\frac{1}{r_0}-\frac{fM}{r_0^2v_0^2\sin^2\alpha}\\&B=-\frac{1}{r_0\tan\alpha}\end{aligned}. \tag{13.61}$$

The equation of motion is obtained by replacing the two constants in (13.58):

$$\frac{1}{r}=\left(\frac{1}{r_0}-\frac{fM}{r_0^2v_0^2\sin^2\alpha}\right)\cos\theta-\frac{1}{r_0\tan\alpha}\sin\theta+\frac{fM}{C^2}, \tag{13.62}$$

or

$$r=\frac{\dfrac{C^2}{fM}}{1+\dfrac{C^2}{fM}\left[\left(\dfrac{1}{r_0}-\dfrac{fM}{C^2}\right)\cos\theta-\dfrac{1}{r_0\tan\alpha}\sin\theta\right]}=\frac{p}{1+e\cos\left(\theta-\theta_0\right)}. \tag{13.63}$$

By simple identification

$$p = \frac{C^2}{fM}$$

$$e\cos\theta_0 = \frac{C^2}{fM}\left(\frac{1}{r_0} - \frac{fM}{C^2}\right); \tag{13.64}$$

$$e\sin\theta_0 = -\frac{C^2}{fM}\frac{1}{r_0\tan\alpha}$$

from which

$$e = \frac{C^2}{fM}\sqrt{\left(\frac{1}{r_0} - \frac{fM}{C^2}\right)^2 + \frac{1}{\left(r_0\tan\alpha\right)^2}}$$

$$\theta_0 = \arctan\left[\frac{r_0\tan\alpha}{\frac{fM}{C^2} - \frac{1}{r_0}}\right]; \tag{13.65}$$

called respectively eccentricity (e) and parameter ($\theta_0$) of a conic.
The conclusion is that the path of a material point in a central force field is **a conic**:

$$r = \frac{p}{1 + e\cos\left(\theta - \theta_0\right)}. \tag{13.66}$$

From the first expression of (13.65) the condition $e=1$ implies for the initial velocity:

$$v_0 = \sqrt{\frac{2fM}{r_0}}. \tag{13.67}$$

From analytic geometry, the nature of the conic depends on the eccentricity:

$$\begin{cases} |e| < 1 \Rightarrow v_0 < \sqrt{\frac{2fM}{r_0}} \Rightarrow \ Ellipse \\ e = 1 \ \ \Rightarrow v_0 = \sqrt{\frac{2fM}{r_0}} \Rightarrow Parabola \\ |e| > 1 \ \Rightarrow v_0 > \sqrt{\frac{2fM}{r_0}} \Rightarrow Hyperbola \end{cases} \tag{13.68}$$

and is independent of the initial angle α. The ellipse becomes a circle if $e=0$.
For an ellipse, two important cases are defined:

a) If $0<e<1$, then for $\theta = 0$ there is a closest point to O called **perigee**.
b) If $-1<e<0$, then for $\theta = 0$ there is a farthest point to O called **apogee.** This kind of ellipse is called **subcircular**.

It has been proven that the nature of the path is independent of $\alpha$. This suggests a particular choice of the Ox axis with respect to the path, such that for $\theta=0$ the velocity is perpendicular on the Ox axis ($\alpha=\pi/2$). In these conditions (13.61) becomes:

$$A = \frac{1}{r_0} - \frac{fM}{r_0^2 v_0^2} \qquad B = 0 \tag{13.69}$$

and the formula for the eccentricity becomes

$$e = \frac{v_0^2 r_0^2}{fM}\left(\frac{1}{r_0} - \frac{fM}{v_0^2 r_0^2}\right), \tag{13.70}$$

so that the path can be expressed as

$$r = \frac{\dfrac{r_0^2 v_0^2}{fM}}{1 + \dfrac{r_0^2 v_0^2}{fM}\left(\dfrac{1}{r_0} - \dfrac{fM}{r_0^2 v_0^2}\right)\cos\theta} = \frac{\dfrac{r_0^2 v_0^2}{fM}}{1 + e\cos\theta}. \tag{13.71}$$

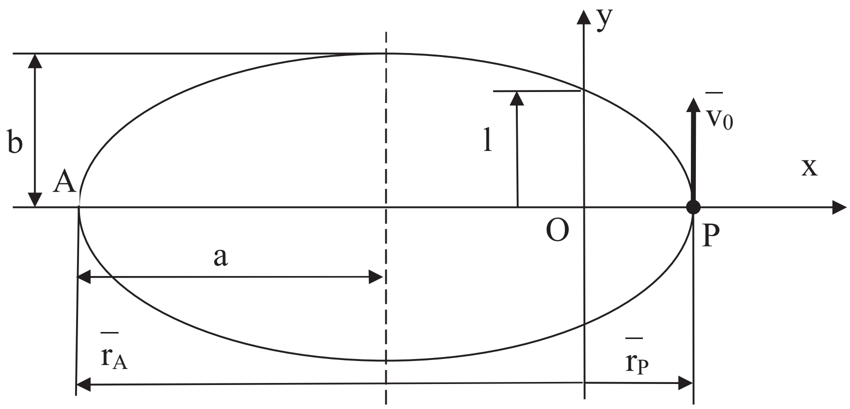



**Fig. 13.8 Elliptic path in symmetric frame. P=Perigee; A=Apogee; O=focus**

The distance between the focus O and the crossing points to the Oy axis are called **semilatus rectum** and are obtained from (13.71) for $\theta=\pi/2$:

$$l = \frac{r_0^2 v_0^2}{fM}. \tag{13.72}$$

Using this distance, the distances to the perigee (OP) and respectively to the apogee (OA) can be written:

$$r_P = r_{(\theta=0)} = \frac{l}{1+e}; \quad r_A = r_{(\theta=\pi)} = \frac{l}{1-e} \tag{13.73}$$

The ellipse semi-major axis is then $a = \frac{r_P + r_A}{2} = \frac{l}{1-e^2}$. By definition the ellipse eccentricity is $e^2 = a^2 - b^2$, so that the semi-minor axis becomes $b = a\sqrt{1-e^2} = \frac{l}{\sqrt{1-e^2}}$. By definition the area of the ellipse is

$$A = \pi ab = \pi \frac{l}{1-e^2}\frac{l}{\sqrt{1-e^2}} = \frac{\pi l^2}{\left(1-e^2\right)^{\frac{3}{2}}} = \pi a^2\sqrt{1-e^2} \tag{13.74}$$

First cosmic velocity

An Earth's satellite is an object having an elliptic orbit with the Earth in one focus. From the eccentricity formula (13.70)

$$e = \frac{v_0^2 r_0}{fM} - 1. \tag{13.75}$$

It follows that a minimum velocity ($v_1$) exist for $e=0$, which corresponds to the circular path. Consequently

$$v_0 = v_1 = \sqrt{\frac{fM}{r_0}}. \tag{13.76}$$

A circular path is considered close to the Earth's surface of radius $r_0=6370$ km (tens or even hundreds of kilometers are a negligible height). The universal attraction force is responsible for the weight of the bodies on the Earth's surface, so

$$mg = f\frac{mM}{r_0^2}. \tag{13.77}$$

For the Earth the product $fM = gr_0^2 = 9.81\cdot 6370000^2 = 3.98\cdot 10^{14}$ $\mathrm{m}^3 s^{-2}$ and from (13.76) can be deduced the **first cosmic velocity**

$$v_1 = \sqrt{gr_0} = 7.905 \text{ km/s}. \tag{13.78}$$

Second cosmic velocity

The minimum velocity required for a satellite to leave the Earth is the velocity necessary to pass from elliptic (or circular) to a parabolic path. From (13.68) this **second cosmic velocity** can be written as:

$$v_2 = \sqrt{\frac{2fM}{r_0}} = \sqrt{2}v_1 = 11.179 \text{ km/s}. \tag{13.79}$$

It has to be mentioned that the Earth is assumed to be fixed in the above solution.

## *13.5. Theorems of Dynamics for a Material Point*

### 13.5.1. Momentum of a material point

Definition

The **momentum** or **linear momentum** of a material point of mass *m*, moving with velocity $\bar{v}$ is defined by the vector:

$$\bar{H} = m\bar{v} . \tag{13.80}$$

Theorem of momentum

The derivative of momentum with respect to time equals the force acting on the material point:

$$\frac{d\bar{H}}{dt} = \bar{F} . \tag{13.81}$$

Proof

$$\frac{d\bar{H}}{dt} = \frac{d(m\bar{v})}{dt} = m\frac{d(\bar{v})}{dt} = m\bar{a} = \bar{F} . \tag{13.82}$$

It has been taken into account that the mass of the material point is a constant in Newtonian Mechanics and use was made of the Second principle of Mechanics ($\bar{F} = m\bar{a}$).

### 13.5.2. Angular momentum of a material point

Definition

The **moment of momentum** or **angular momentum** about a fixed point O, of a material point *M* of mass *m* moving with velocity $\bar{v}$ is by definition the vector product:

$$\bar{K}_O = \bar{r} \times m\bar{v} \tag{13.83}$$

Theorem of angular momentum

The derivative of angular momentum about a fixed point O with respect to time equals the moment about the same point O of the force acting on the material point:

$$\frac{d\bar{K}_O}{dt} = \bar{M}_O(\bar{F}). \tag{13.84}$$

Proof

$$\frac{d\bar{K}_O}{dt} = \frac{d}{dt}(\bar{r} \times m\bar{v}) = \frac{d\bar{r}}{dt} \times m\bar{v} + \bar{r} \times m\frac{d\bar{v}}{dt} = \bar{v} \times m\bar{v} + \bar{r} \times m\bar{a} = \bar{r} \times \bar{F} = \bar{M}_O(\bar{F}) \tag{13.85}$$

It has been taken into account that the mass of the material point is a constant in Newtonian Mechanics and also $\overline{v} \times m\overline{v} = \overline{0}$ and $\overline{F} = m\overline{a}$.

### 13.5.3. Kinetic energy and work for a material point

Definitions

Kinetic energy for a material point of mass $m$ moving with velocity $\overline{v}$ is defined by:

$$T = \frac{1}{2} m v^2 \tag{13.86}$$

The elementary work of a force $\overline{F}(X, Y, Z)$ acting on a material point is defined by the scalar product:

$$dW = \overline{F} \cdot d\overline{r} = Xdx + Ydy + Zdz \tag{13.87}$$

Theorem

The changing rate of the kinetic energy for material point equals the elementary work of the force acting on this material point

$$dT = dW \tag{13.88}$$

Proof

$$\begin{aligned} dW &= \overline{F} \cdot d\overline{r} = m\overline{a} \cdot d\overline{r}(t) = m\frac{d\overline{v}}{dt} \cdot \frac{d\overline{r}}{dt} dt = m\frac{d\overline{v}}{dt} dt \cdot \frac{d\overline{r}}{dt} = md\overline{v} \cdot \overline{v} \\ &= md\left(\frac{\overline{v} \cdot \overline{v}}{2}\right) = md\left(\frac{v^2}{2}\right) = d\left(m\frac{v^2}{2}\right) = dT \end{aligned} \tag{13.89}$$

### 13.5.4. Theorems of Conservation

a) The linear momentum of a material point is constant, if and only if $\overline{F} = \overline{0}$.

   Indeed $\frac{d\overline{H}}{dt} = \overline{0} \Leftrightarrow \overline{H} = const.$

b) The projection on an axis of the momentum for a material point is constant if the projection of the force $\overline{F}$ on this axis is equal to zero.

   For example, if X = 0 then $\frac{dH_x}{dt} = 0 \Leftrightarrow H_x = mv_x = const.$

c) The angular momentum of a material point about the fixed point O is constant, if and only if the moment of F about O is equal to zero:

   Indeed $\frac{d\overline{K}_O}{dt} = \overline{0} \Leftrightarrow \overline{K}_O = const.$

d) The angular momentum of a material point about a fixed axis Δ is constant if the moment of $\bar{F}$ about Δ is null.
For example the moment about Oz axis is
$M_{Oz} = xY - yX = 0 \Leftrightarrow K_{Oz} = m\left(xv_y - yv_x\right) = const.$
If the velocity is expressed in cylindrical coordinates (polar radius $r_P$, polar angle $\theta$ and height $z$), $\bar{v} = \dot{r}_P\bar{\rho} + r_P\dot{\theta}\bar{n} + \dot{z}\bar{k}$ and the position vector is $\bar{r} = r_P\bar{\rho} + z\bar{k}$ , then the angular momentum can be written as

$$\bar{K}_O = \bar{r} \times m\bar{v} = m\begin{vmatrix} \bar{\rho} & \bar{n} & \bar{k} \\ r_P & 0 & z \\ \dot{r}_P & r_P\dot{\theta} & \dot{z} \end{vmatrix} = m\left[-zr_P\dot{\theta}\bar{\rho} + \left(\dot{r}_P z - \dot{z}r_P\right)\bar{n} + r_P^2\dot{\theta}\bar{k}\right].$$

Assuming again that the moment about the Oz axis is null ($M_{Oz}= 0$), then $K_{Oz} = mr_P^2\dot{\theta} = 2m\Omega = const.$
It follows that the projection of M on a plane perpendicular to the axis Δ moves with a constant area velocity (rate of sweeping out area), about the intersection point of the axis Δ to the plane.

### 13.5.5. Theorem of kinetic energy and work based on the force function

In general, the elementary work of a force $\bar{F}$ is not an exact differential $dU\,(x, y, z)$. For this reason, the elementary work has been denoted by $đW$ and not as usual by $dW$.

If $dW$ is an exact differential $dU(x, y, z)$ then:

$$\bar{F}d\bar{r} = Xdx + Ydy + Zdz = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\,. \tag{13.90}$$

It would be useful to conclude that:

$$X = \frac{\partial U}{\partial x}; \quad Y = \frac{\partial U}{\partial y}; \quad Z = \frac{\partial U}{\partial z}, \tag{13.91}$$

but these relations hold if and only if:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}; \quad \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}; \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}\,. \tag{13.92}$$

The force components derive thus from U, which is called for this reason a **force function**. If these relations are satisfied, then the force function can be determined from the following contour integrals between state ‘0’ and ‘1’, the integral being independent on the integration path:

$$U(x,y,z)=\oint_{x_0\to x_1} X(x,y,z)dx+\oint_{y_0\to y_1} Y(x,y,z)dy+\oint_{z_0\to z_1} Z(x,y,z)dz\,. \quad (13.93)$$

In these conditions, the theorem of kinetic energy and elementary work (13.88) can be cast into the form "The sum of the kinetic and potential energy is constant":

$$dT=dU \;\Leftrightarrow\; d(T-U)=0 \;\Leftrightarrow T-U=E \;\Leftrightarrow\; T+V=E\,. \quad (13.94)$$

The following notations have been used:

- $V$ is called **potential energy** ($V=-U$) and
- $E$ is called the **total mechanical energy** and is a constant if all forces applied on a material point can be derived from force functions $U_1$, $U_2$, ...

This theorem is called also the principle of the conservation of energy.

The principle expressed mathematically by (13.94) is one of the fundamental formulas of Mechanics, and it is largely used in the solution of problems. However it should be noted that this theorem is valid if and only if, the field of forces is a conservative one, because in general the work $W$ is a contour integral:

$$W=\oint_{0-1} Xdx+Ydy+Zdz\,. \quad (13.95)$$

Its value depends on the path of the material point from state '0' to state '1' and thus is unknown 'a priori' since in general the path of the material point is to be determined in the majority of problems concerning particle dynamics.

In these conditions, the theorem of energy and work can be written in a finite form as:

$$T-T_0=W=\oint_{0-1} Xdx+Ydy+Zdz\,. \quad (13.96)$$

### 13.5.6. Examples of conservative fields

a) The gravity constant field

A heavy material point is acted by a force whose components are $X = 0$, $Y = 0$, $Z = -mg$ (Fig. 13.9). The conditions (13.92) are accomplished. Applying the (13.93) formula, it follows that

$$U=-mg\oint_{0\to 1} dz=-mgz_1+mgz_0=mg(z_0-z_1) \quad (13.97)$$

As a consequence:

$$W_{0\to 1}=-V=U=mg\Delta h\,. \quad (13.98)$$

A positive work corresponds to a descending motion ($\Delta h=z_0-z_1>0$), $\Delta h$ being the level difference between the initial and the final position on the path.

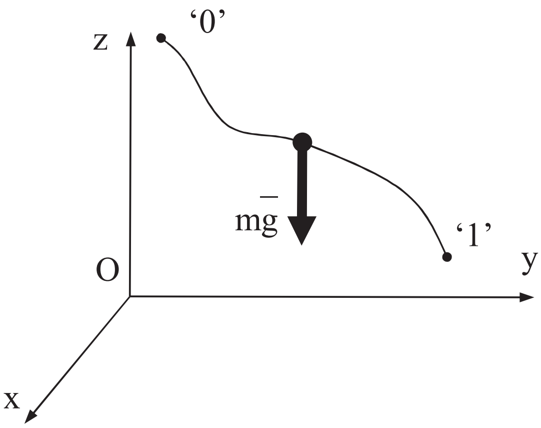



**Fig. 13.9 A particle moving in a constant gravitational field**

b) <u>The elastic force</u>

The elastic force has the expression $\overline{F} = -k\overline{r}$ , in which k is the elastic constant, measured in N/m and the origin for the position vector $\overline{r}$ is a position of free form of the elastic spring.

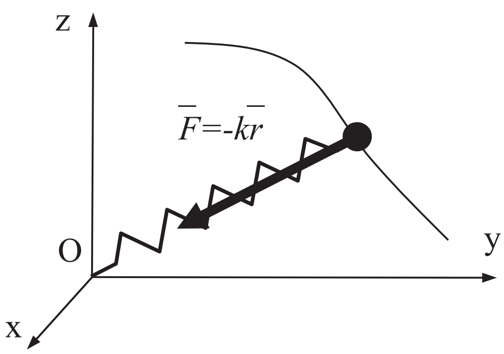



**Fig. 13.10 A moving particle acted by an elastic force**

The force projections are $X = -kx;\ \ Y = -ky;\ \ Z = -kz$. The conditions (13.92) are verified. Applying the (13.93) formula, it follows that:

$$U = \oint_{0-1} Xdx + Ydy + Zdz = -\oint_{0-1} kxdx - \oint_{0-1} kydy - \oint_{0-1} kzdz = -\frac{k}{2}\left(x^2 + y^2 + z^2\right). \quad (13.99)$$

Since $\left|\overline{F}\right| = k\left|\overline{r}\right| = k\sqrt{x^2 + y^2 + z^2}$ it follows that the potential energy V has the expression:

$$V = -U = -W = \frac{1}{2}\left|\overline{F}\right|\left|\overline{r}\right|, \quad (13.100)$$

where $\left|\overline{r}\right|$ represents the deformation of the spring.

c) <u>The universal attraction field</u>

The Newtonian gravitational force produced by the universal attraction has the expression:

$$\overline{F} = -f\frac{mM}{r^2}\frac{\overline{r}}{|\overline{r}|} = -f\frac{mM}{r^3}\overline{r} = -\frac{fmM}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\left(x\overline{i}+y\overline{j}+z\overline{k}\right) \quad (13.101)$$

The compatibility conditions are in this case also fulfilled. By integration between states '0' and '1', the force function can be written

$$U = -fmM\left[\oint_{x_0\to x_1}\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dx + \oint_{y_0\to y_1}\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dy + \oint_{z_0\to z_1}\frac{z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dz\right] \quad (13.102)$$

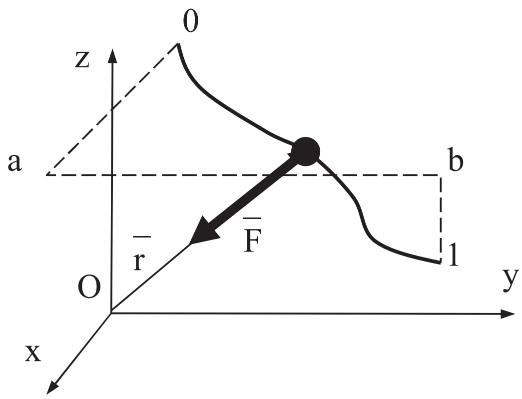



**Fig. 13.11 Particle moving in a universal attraction field of forces**

The integrals are independent of path, so instead of the effective path 0-1, can be chosen 0-a (changing only x), then a-b (changing only y) and finally b-1 (changing only z). The integrals are:

$$\begin{aligned}
&\oint_{0\to a}\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dx = -\frac{1}{\left(x_a^2+y_0^2+z_0^2\right)^{\frac{1}{2}}} + \frac{1}{\left(x_0^2+y_0^2+z_0^2\right)^{\frac{1}{2}}}\\
&\oint_{a\to b}\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dy = -\frac{1}{\left(x_a^2+y_b^2+z_0^2\right)^{\frac{1}{2}}} + \frac{1}{\left(x_a^2+y_0^2+z_0^2\right)^{\frac{1}{2}}}. \quad (13.103)\\
&\oint_{b\to 1}\frac{z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dz = -\frac{1}{\left(x_a^2+y_b^2+z_1^2\right)^{\frac{1}{2}}} + \frac{1}{\left(x_a^2+y_b^2+z_0^2\right)^{\frac{1}{2}}}
\end{aligned}$$

After injecting these results in(13.102), the force function becomes:

$$
\begin{aligned}
U &= fmM\left[\frac{1}{\left(x_a^2+y_b^2+z_1^2\right)^{\frac{1}{2}}}-\frac{1}{\left(x_0^2+y_0^2+z_0^2\right)^{\frac{1}{2}}}\right] \\
&= fmM\left[\frac{1}{\left(x_1^2+y_1^2+z_1^2\right)^{\frac{1}{2}}}-\frac{1}{\left(x_0^2+y_0^2+z_0^2\right)^{\frac{1}{2}}}\right]=fmM\left[\frac{1}{r_1}-\frac{1}{r_0}\right]
\end{aligned}
\quad . \tag{13.104}
$$

The state '0' can be taken at $r_0 \to \infty$, so that for a current position *r*, the potential energy *V*= -*U* can be written

$$V=-f\frac{mM}{r}. \tag{13.105}$$

### *13.6. Dynamics of the constrained material point*

A material point is subjected to a given force $\overline{F}$ and is constrained to move on a given curve or on a given rigid surface. Applying the axiom of constraints (§ 5.3 – vol.1) the constraint can be replaced by a force $\overline{R}$, called reaction. In this manner the dynamic problem of a constrained material point is reduced to the dynamics of a free material acted on by two forces: the given force $\overline{F}$ and the reaction $\overline{R}$. The fundamental equation of dynamics can be written:

$$m\overline{a}=\overline{F}\left(\overline{r},\overline{v},t\right)+\overline{R}\left(\overline{r},\overline{v},t\right) \tag{13.106}$$

In general, computing the motion of the material point implies the elimination of the reaction force $\overline{R}$. One possibility is to apply the theorem of linear momentum on an axis perpendicular to $\overline{R}$, or the theorem of angular momentum about an axis, about which $\overline{R}$ has no moment component, or otherwise, applying the theorem of energy if the work of the reaction $\overline{R}$ is obviously null. This later situation appears if the curves or surfaces are smooth.

#### 13.6.1. Examples

a) Mathematical pendulum

A mathematical pendulum is a material point attached to a fixed point by a weightless and rigid truss or by a weightless and inextensible string. It is moving in a vertical plane being subjected to the gravitational force. If a truss is used, the constraint is called **bilateral**; if a string is used, the constraint is **unilateral** because only tension can exist in a string.

The length of the pendulum is *l*, the initial position $M_0$, a certain position is defined by the angle θ (Fig. 13.12).

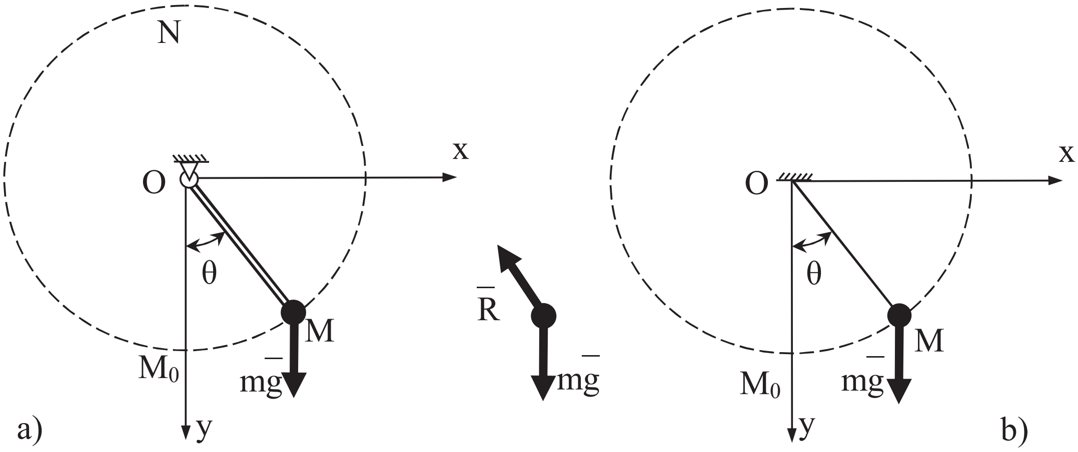



**Fig. 13.12 Mathematical pendulum with bilateral (a) or unilateral (b) constraint**

The forces acting on the material point are the weight and the reaction force so, the fundamental equation of dynamics can be written: $m\overline{a} = m\overline{g} + \overline{R}$. Projections of this equation on a polar coordinate frame with polar radius along OM are:

$$\begin{aligned} m\left(\ddot{l} - l\dot{\theta}^2\right) &= mg\cos\theta - R \\ m\left(l\ddot{\theta} + 2\dot{l}\dot{\theta}\right) &= -mg\sin\theta \end{aligned} \tag{13.107}$$

The second equation completely defines the motion:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{13.108}$$

The application of the energy and work theorem with: $T_0 = \frac{1}{2}mv_0^2$; $T_1 = \frac{1}{2}mv^2$, $W = -mgl\left(1 - \cos\theta\right)$ corresponds to:

$$\frac{1}{2}m\left(v^2 - v_0^2\right) = -mgl\left(1 - \cos\theta\right). \tag{13.109}$$

It follows that:

$$v^2 = v_0^2 - 2gl\left(1 - \cos\theta\right). \tag{13.110}$$

This theorem is clearly more advantageous. However the same result can be obtained by the following calculus, starting from (13.108):

$$\ddot{\theta}\dot{\theta} + \frac{g}{l}\dot{\theta}\sin\theta = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) - \frac{d}{dt}\left(\frac{g}{l}\cos\theta\right) = 0 \tag{13.111}$$

which means that the constant functions multiplied by $l^2$ are:

$$l^2\frac{\dot{\theta}^2}{2} - gl\cos\theta = l^2\frac{\dot{\theta}_0^2}{2} - gl\cos\theta_0. \tag{13.112}$$

But $\theta_0 = 0$; $l\dot{\theta} = v$; $l\dot{\theta}_0 = v_0$, so that after simplifications, the same expression as (13.110) is obtained. The obtained result is valid for both types of constraints. A separation is necessary in the following analysis, based on the type of constraint.

Case of bilateral constraint

- The mathematical pendulum is **oscillating** if, there exists an angle $\theta = \alpha < 180^o$, for which $v = 0$.

Imposing $v = 0$ in (13.110), it follows

$$\cos\theta = 1 - \frac{v_0^2}{2gl}, \tag{13.113}$$

and the condition for a real solution is $|\cos\theta| \le 1$. The upper limit is clearly verified, but the lower limit leads to

$$v_0^2 = 4gl \;\Rightarrow\; \alpha = 180^\circ. \tag{13.114}$$

The point N ($\alpha = 180^\circ$), the highest point on the circle, is a position of equilibrium. If the material point is placed at N, it remains at rest. The initial velocity given by (13.114) is in principle necessary to the particle to arrive at N. However the material point will never reach this point, because the time required to do so, is infinite:

$$\begin{aligned} t &= \int_M^N dt = \int_0^{\pi l} \frac{ds}{v} = \int_0^{\pi} \frac{l\,d\theta}{\sqrt{v_0^2 - 2gl(1-\cos\theta)}} = \int_0^{\pi} \frac{l\,d\theta}{\sqrt{2gl(1+\cos\theta)}} \\ &= \frac{1}{2}\sqrt{\frac{l}{g}} \int_0^{\pi} \frac{d\theta}{\sqrt{\cos^2\frac{\theta}{2}}} = \frac{1}{2}\sqrt{\frac{l}{g}} \int_0^{\pi} \frac{d\theta}{\left|\cos\frac{\theta}{2}\right|} = \sqrt{\frac{l}{g}} \left[ \ln\left|\tan\left(\frac{\theta}{4} + \frac{\pi}{4}\right)\right| \right]_0^{\pi} = \infty \end{aligned} \tag{13.115}$$

- The motion of the mathematical pendulum has an **asymptotic** nature.
- If $v_0^2 > 4gl$, there is a **circular motion**, because the particle will move beyond N and will return towards M, continuing indefinitely this motion.

Resuming the above results, the motions taking place for bilateral constraint are, for

- $v_0^2 < 4gl$, oscillating motion;
- $v_0^2 = 4gl$, asymptotic motion;
- $v_0^2 > 4gl$, rotating motion.

Case of unilateral constraint

In this case, the particle can leave the circle if the reaction force $R=0$. The projection of the fundamental equation of dynamics on the normal to the circle can be written (13.107)

$$-\frac{mv^2}{l} = mg\cos\theta - R\,. \tag{13.116}$$

If $v$ is replaced by its expression(13.110), the reaction R becomes:

$$R = \frac{m}{l}\left[v_0^2 - 2gl + 3gl\cos\theta\right]. \tag{13.117}$$

The reaction possibly vanishes for an angle β:

$$\cos\theta = \frac{2gl - v_0^2}{3gl}. \tag{13.118}$$

The existence condition for a real angle $\theta$ is $|cos\theta|\leq 1$.The lower limit imposes $v_0^2 \leq 5gl$ and corresponds to angles $\theta>\pi/2$ (negative values for cos $\theta$) or from (13.118) $v_0^2 > 2gl$.

It can be inferred that for given initial velocity, three cases can exist:

- $v_0^2 < 2gl$, oscillating motion;
- $2gl \leq v_0^2 \leq 5gl$, the particle leaves the circle at an angle $\beta$ given by (13.118);
- $v_0^2 > 5gl$, circular motion of indefinite duration.

It can be easily remarked that for unilateral constraint the rotation is obtained for higher initial velocity ($v_0$>$5gl$) than for unilateral constraint ($v_0$>$4gl$).

b) Spherical pendulum

The spherical pendulum, contrary to the mathematical pendulum is not restricted to move in a vertical plane. Only the distance between the particle of mass $m$ and a fixed point O remains a constant denoted $R$ during the motion.

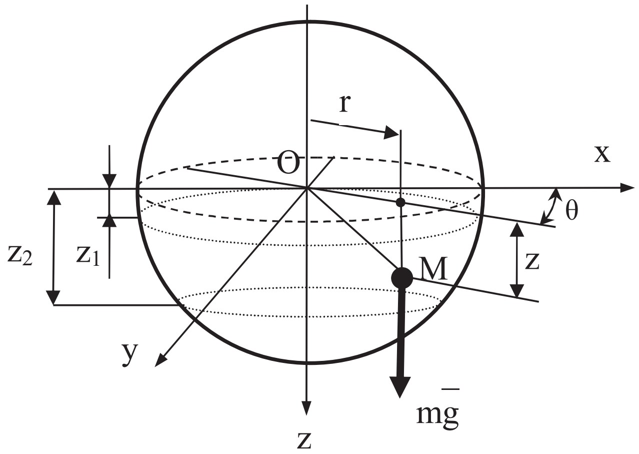



**Fig. 13.13 Spherical pendulum**

If θ and z are the curvilinear (cylindrical) coordinates of a point M on the sphere of radius R, the position of the point M can be expressed in a fixed Cartesian frame as:

$$x=\sqrt{R^2-z^2}\cos\theta;\quad y=\sqrt{R^2-z^2}\sin\theta;\quad z=z; \tag{13.119}$$

The velocity of M has the following projections:

$$\begin{aligned}\dot{x}&=\frac{-z\dot{z}}{\sqrt{R^2-z^2}}\cos\theta-\sqrt{R^2-z^2}\dot{\theta}\sin\theta\\ \dot{y}&=\frac{-z\dot{z}}{\sqrt{R^2-z^2}}\sin\theta+\sqrt{R^2-z^2}\dot{\theta}\cos\theta\\ \dot{z}&=\dot{z}\end{aligned} \tag{13.120}$$

The functions *θ(t)* and *z(t)* can be expressed using the theorem of the angular momentum projected on the Oz axis and the theorem of energy and work. Since

$$\overline{K}_O=\overline{r}\times m\overline{v}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ x & y & z\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix}=m\left(y\dot{z}-z\dot{x}\right)\overline{i}+m\left(z\dot{x}-x\dot{z}\right)\overline{j}+m\left(x\dot{y}-y\dot{x}\right)\overline{k}, \tag{13.121}$$

the theorem of angular momentum about the Oz axis is $\frac{d}{dz}\left(K_{Oz}\right)=M_{Oz}$ in which $M_{Oz}$ represents the moments about the Oz axis of all the forces applied on M. Since the weight is parallel to the Oz axis and the normal reaction crosses the Oz axis, $M_{Oz}=0$ and the theorem states that $K_{Oz}=C=constant.$, *i.e.:*

$$m\left(R^2-z^2\right)\dot{\theta}=mC_1=const. \tag{13.122}$$

The theorem of kinetic energy and work states that $T-T_0=W_{01}$:

$$\frac{1}{2}m\left[\frac{z^2\dot{z}^2}{R^2-z^2}+\left(R^2-z^2\right)\dot{\theta}^2\right]-\frac{1}{2}mv_0^2=mg\left(z-z_0\right) \tag{13.123}$$

Replacing $\dot{\theta}$ from (13.122) in the last equation, the following expression is obtained:

$$\frac{z^2\dot{z}^2+C_1^2}{R^2-z^2}-v_0^2=2g\left(z-z_0\right). \tag{13.124}$$

The vertical component of the velocity is then:

$$\dot{z}=\pm\frac{1}{z}\sqrt{\left(R^2-z^2\right)\left[2g\left(z-z_0\right)+v_0^2\right]-C_1^2}=\pm\frac{1}{z}\sqrt{P_3\left(z\right)}, \tag{13.125}$$

in which

$$P_3\left(z\right)=\left(R^2-z^2\right)\left[2g\left(z-z_0\right)+v_0^2\right]-C_1^2. \tag{13.126}$$

Investigating the signs of the polynomial in the following table, it follows that there are three real roots, from which two ($z_1$ and $z_2$) lie in the interval [-R, R].
It is known that for the points on the sphere $R > z_0$ and in general $v_0^2 = \frac{z_0^2 \dot{z}_0^2}{R^2 - z_0^2} + \left(R^2 - z_0^2\right)\dot{\theta}_0^2 > \left(R^2 - z_0^2\right)\dot{\theta}_0^2$, so that:

$$P_3(z_0) = \left(R^2 - z_0^2\right)v_0^2 - \left(R^2 - z^2\right)^2 \dot{\theta}_0^2 = \left(R^2 - z_0^2\right)\left[v_0^2 - \left(R^2 - z_0^2\right)\dot{\theta}_0^2\right] > 0.$$

| z | -∞ | -R | $z_0$ | R | ∞ |
|---|---|---|---|---|---|
| $P_3(z)$ | -∞ | $-C_1^2<0$ | >0 | $-C_1^2<0$ | ∞ |

The spherical pendulum winds in general between two planes defined by $z = z_1$ and $z = z_2$. Particular cases are motions on vertical or horizontal circles.

### *13.7. Dynamics of the Relative Motion of a Material Point*

Newton's second principle is valid only for a fixed frame. If the motion is defined in a frame moving relative to an assumed fixed frame:

$$\bar{a}_r = \bar{a}_a - \bar{a}_t - \bar{a}_c, \tag{13.127}$$

where $\bar{a}_a, \bar{a}_r, \bar{a}_t, \bar{a}_c$ denote respectively the accelerations: absolute, relative, of transport and Coriolis' acceleration. Multiplying this expression on both sides by the mass *m* of the given material point, one gets:

$$m\bar{a}_r = m\bar{a}_a - m\bar{a}_t - m\bar{a}_c. \tag{13.128}$$

According to Newton's second principle $m\bar{a}_a = \bar{F}$, for the force acting on the given material point, it will be denoted:

$$\bar{F}_t = -m\bar{a}_t; \qquad \bar{F}_c = -m\bar{a}_c. \tag{13.129}$$

The vector $\bar{F}_t$ is called the force of transport and the vector $\bar{F}_c$ is called the Coriolis force. It should be noted that the vectors $\bar{F}_t$ and $\bar{F}_c$ do not represent real forces because there is no material system acting on the given material point with these forces. These are called sometimes "**pseudo-forces**" or "**inertia forces**" only for practical reasons. In fact $\bar{F}_t$ and $\bar{F}_c$ are correction terms for Newton's second principle. The fundamental equation of the dynamics of the relative motion of a material point is then

$$m\bar{a}_r = \bar{F} + \bar{F}_t + \bar{F}_c. \tag{13.130}$$

where the complete expressions of the inertia forces are

$$\begin{aligned} \bar{F}_t &= -m\bar{a}_t = -m\left[\bar{a}_0 + \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times \left(\bar{\omega} \times \bar{r}\right)\right] \\ \bar{F}_c &= -m\bar{a}_c = -2m\bar{\omega} \times \bar{v}_r \end{aligned}. \tag{13.131}$$

It is reminded here that $\overline{a}_0, \overline{\omega}, \overline{\varepsilon}$ are the acceleration of the origin of the moving frame, its angular velocity and acceleration respectively and $\overline{v}_r$ is the velocity of the material point relative to the moving frame.

### *13.8. Inertial Frames*

Newton's absolute and immovable space could not be identified. A moving frame, for which Newton's second principle is valid, is called an **inertial frame**.
It is necessary that $\overline{F}_t + \overline{F}_c = \overline{0}$ for any motion and one possibility is

$$m\left[\overline{a}_0 + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times \left(\overline{\omega} \times \overline{r}\right)\right] + 2m\overline{\omega} \times \overline{v}_r = \overline{0}. \tag{13.132}$$

By considering two motions having at the time moment *t* the same position vector $\overline{r}$ and arbitrary relative velocities $\overline{v}'$ and $\overline{v}''$, the previous equation yield

$$\begin{gathered} m\left[\overline{a}_0 + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times \left(\overline{\omega} \times \overline{r}\right)\right] + 2m\overline{\omega} \times \overline{v}_r' = \overline{0} \\ m\left[\overline{a}_0 + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times \left(\overline{\omega} \times \overline{r}\right)\right] + 2m\overline{\omega} \times \overline{v}_r'' = \overline{0} \end{gathered}. \tag{13.133}$$

By subtracting these equations, it follows that

$$2m\overline{\omega} \times \left(\overline{v}_r' - \overline{v}_r''\right) = \overline{0}. \tag{13.134}$$

Since $\overline{v}'$ and $\overline{v}''$ are arbitrary, it follows that $\overline{\omega} = \overline{0}$. The angular acceleration is then $\overline{\varepsilon} = \dot{\overline{\omega}} = \overline{0}$. From the equation (13.132) the acceleration $\overline{a}_0 = \overline{0}$.

It can be concluded that an inertial frame has a uniform rectilinear translation with respect to Newton's absolute and immovable space.

One important consequence can be stated: the uniform rectilinear translation of a reference frame cannot be determined by any mechanical experiment. This is the principle of relativity in classical mechanics. Albert Einstein has generalized this principle for any physical experiment.

### *13.9. Relative Rest*

If a material point is at rest relative to a moving frame, then its relative acceleration $\overline{a}_r = \overline{0}$ and relative velocity $\overline{v}_r = \overline{0}$. It follows hence that Coriolis acceleration $\overline{a}_c = \overline{0}$ is also null, and with it the Coriolis force $\overline{F}_c = \overline{0}$. From equation (13.130) can therefore be obtained:

$$\overline{0} = \overline{F} + \overline{F}_t \tag{13.135}$$

It can be stated that a material point is at relative rest, if the applied force(s) F is(are) in equilibrium with the force of transport.

**Example**

Determine the relative motion of a particle along a rod having a uniform rotation of angular velocity $\overline{\omega}$ about the $Oz_0$ fixed axis. At the initial moment, $x_{t=0} = x_0$ and $\dot{x}_{t=0} = 0$ and the angle between the rod and the vertical $Oz_0$ axis is of constant value α.

A moving frame can be linked to the rod with Ox axis along the rod and Oy axis permanently included in the $O_0x_0y_0z_0$ plane. It follows that $\overline{a}_0 = \overline{0}$ since the origin is on the fixed axis and $\overline{\varepsilon} = \dot{\overline{\omega}} = \overline{0}$.

The relative position of the particle is $\overline{r} = x\overline{i}$, so that the relative velocity and acceleration become successively $\overline{v}_r = \dot{x}\overline{i}$; $\overline{a}_r = \ddot{x}\overline{i}$.

The transport acceleration

$$\overline{a}_t = \overline{a}_0 + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times \left(\overline{\omega} \times \overline{r}\right) = \omega\overline{k}_0 \times \left(\omega\overline{k}_0 \times x\overline{i}\right) = \omega\overline{k}_0 \times \left(\omega x \sin\alpha\right)\overline{j} = -\omega^2 x \sin\alpha \overline{i}_1$$

in which $\overline{k}_0$ is the unit vector of the $Oz_0$ axis and $\overline{i}_1$ is the unit vector of the projection onto $O_0x_0y_0$ plane of the Ox axis (see Fig. 13.14). This last unit vector makes an angle θ with the fixed axis $O_0x_0$ and remains included in the $O_0x_0y_0z_0$ plane during the motion.

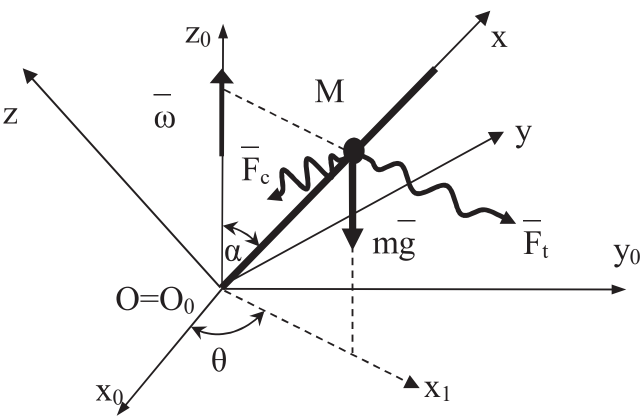



**Fig. 13.14 Relative motion of a particle along a rod**

The associated pseudo-force of transport $\overline{F}_t = m\omega^2 x \sin\alpha\left(\sin\alpha\ \overline{i} - \cos\alpha\ \overline{k}\right)$ is depicted as a wavy line on Fig. 13.14 to emphasize its specificity as pseudo-force.

The Coriolis acceleration is $\overline{a}_c = 2\overline{\omega} \times \overline{v}_r = 2\omega\overline{k}_0 \times \dot{x}\overline{i} = 2\omega\dot{x}\sin\alpha\overline{j}$ and the Coriolis pseudo-force can be written $\overline{F}_c = -2m\omega\dot{x}\sin\alpha\overline{j}$

The differential equations of motion have projections on the moving frame:

$$(Ox)\ m\ddot{x} = -mg\cos\alpha + m\omega^2 x\sin^2\alpha$$

$$(Oy)\ m\ddot{y} \equiv 0 = -2m\omega\dot{x}\sin\alpha + N_y$$

$$(Oz)\ m\ddot{z} \equiv 0 = -m\omega^2 x\sin\alpha\cos\alpha - mg\cos\alpha + N_z$$

The normal reaction between a particle and a material line is normal to the line and this fact has led to the two projections ($N_y$ and $N_z$).

The first differential equation is inhomogeneous of second order: $\ddot{x} - (\omega^2\sin^2\alpha)x = -g\cos\alpha$. The characteristic equation of the homogeneous counterpart is $\lambda^2 = \omega^2\sin^2\alpha$ with two real roots $\lambda_{1,2} = \pm\omega\sin\alpha$. One manner of writing the general solution of the homogeneous equation is $x = C_1\cosh(\omega t\sin\alpha) + C_2\sinh(\omega t\sin\alpha)$. One particular solution of the inhomogeneous equation is $x_p = \dfrac{g\cos\alpha}{\omega^2\sin^2\alpha}$, so that the general solution of the given problem is

$$x = C_1\cosh(\omega t\sin\alpha) + C_2\sinh(\omega t\sin\alpha) + \frac{g\cos\alpha}{\omega^2\sin^2\alpha}.$$

Imposing the initial conditions

$$x_0 = C_1 + \frac{g\cos\alpha}{\omega^2\sin^2\alpha} \qquad \Rightarrow C_1 = x_0 - \frac{g\cos\alpha}{\omega^2\sin^2\alpha};$$

$$0 = C_2\omega\sin\alpha\cosh(\omega 0\sin\alpha) = C_2\omega\sin\alpha \quad \Rightarrow C_2 = 0$$

so that the equation of motion becomes:

$$x = \left(x_0 - \frac{g\cos\alpha}{\omega^2\sin^2\alpha}\right)\cosh(\omega t\sin\alpha) + \frac{g\cos\alpha}{\omega^2\sin^2\alpha}$$

The distance increases as an exponential function. However if $x_0 = \dfrac{g\cos\alpha}{\omega^2\sin^2\alpha}$, it is obvious that for any moment $x = x_0 = \dfrac{g\cos\alpha}{\omega^2\sin^2\alpha}$. This is a position of relative rest.

This position may be determined directly from (13.135) as

$$m\overline{g} + \overline{N}_1 + \overline{N}_2 + \overline{F}_t = \overline{0}$$

The equilibrium must be enforced along the Ox axis: $-mg\cos\alpha + m\omega^2 x\sin^2\alpha = 0$.

It follows that $x = x_0 = \dfrac{g\cos\alpha}{\omega^2\sin^2\alpha} \quad ; \forall t$

## *13.10. Influence of Earth's motion*

Our planet Earth is rotating around its N-S axis and moves around the Sun and with the solar system moves into the galaxy. The correction terms due to these motions are evaluated in order to estimate their influence on a fixed or moving material point.

A frame attached to the Earth's surface Oxyz is moving with the Earth. Two basic motions exist simultaneously: the motion of **revolution** around the Sun and the **rotation** about its North-South (N-S) axis.
The Earth revolves around the Sun in 365 days and 5 hours or 8765 hours, along an ellipse with low eccentricity (minimum/maximum distance Earth-Sun: 146/152 million km), included in the **ecliptic plane,** the geometric plane containing the mean orbit of the Earth around the Sun. Its N-S axis is tilted by an angle of 23˚27" measured from the perpendicular on the ecliptic and the principal effect is the existence of seasons.

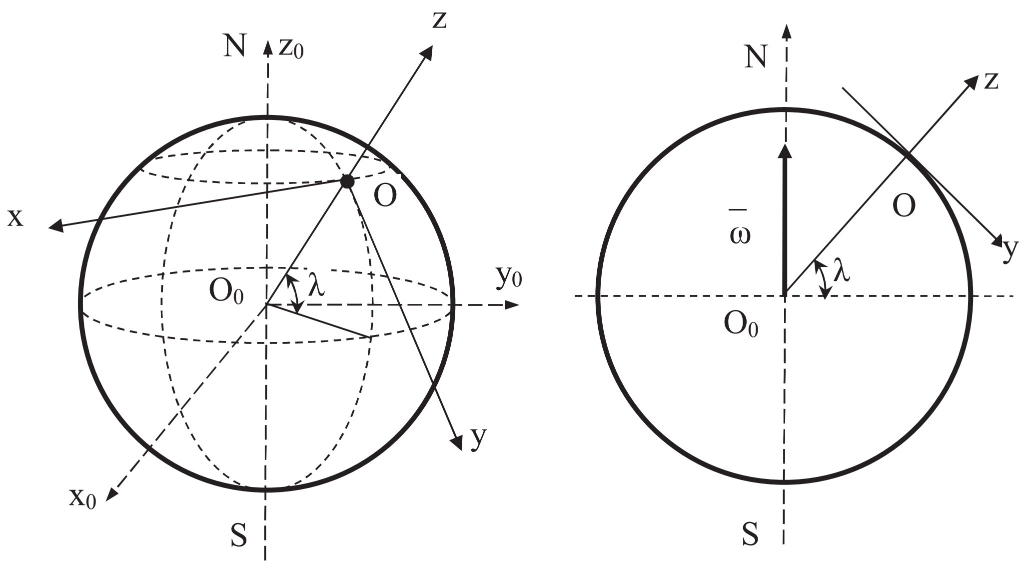



**Fig. 13.15 Earth as a moving frame**

Because of this low eccentricity, the mass center of the Earth can be assumed to have a uniform circular motion around the Sun and its acceleration is:

$$a_{O_0} = \left(\frac{2\pi}{T}\right)^2 R_S = \left(\frac{2\pi}{8765 \cdot 3600}\right)^2 148 \cdot 10^9 = 0.0059\, m/s^2 \qquad (13.136)$$

This acceleration is negligible compared to the mean valued acceleration of gravitation g = 9.81 m/s, so that with sufficient accuracy, the influence of the motion of revolution can be neglected.

### 13.10.1. Equilibrium of a hanging particle on Earth's surface

The rotation of our planet about its N-S axis can be considered to be a uniform rotation ($\overline{\varepsilon} = \overline{0}$), because the variation of its daily rotation duration is negligible. The extremely small variations are due to, among other causes, the slow mass increase caused by falling meteorites.
The frame attached to the Earth has its center O at **latitude** λ (angle measured between the local radius of the Earth and the **equator** plane ($O_0x_0y_0$, Fig. 13.15).

The direction of the local radius of the Earth is taken as Oz axis, the Ox axis is pointing towards west, tangent to the local circle which is parallel to the equator plane. The Oy axis is also tangent to Earth's surface and included in the N-S-O plane, pointing south.
Neglecting the very small quantities discussed above, the Rivals formula for the acceleration of point O of the Earth becomes :

$$\bar{a}_t = \bar{a}_{O_0} + \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) \simeq \bar{\omega} \times (\bar{\omega} \times \bar{r}) \qquad (13.137)$$

The position vector of the point O is $\bar{r} = R\bar{k}$ in which R=6376.5 km is the mean radius of the Earth. The components of the transport acceleration are projected on this frame. Consequently the projections of the angular velocity of the Earth ($|\omega| = \frac{2\pi}{24 \cdot 3600} = 72.722 \cdot 10^{-6} rad/s$) are

$$\bar{\omega} = -\omega \cos\lambda \bar{j} + \omega \sin\lambda \bar{k} . \qquad (13.138)$$

The velocity of the point O is

$$\bar{v}_O = \bar{\omega} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & -\omega\cos\lambda & \omega\sin\lambda \\ 0 & 0 & R \end{vmatrix} = -\omega R \cos\lambda \bar{i} \qquad (13.139)$$

According to (13.137), the acceleration of the origin O of this frame is

$$\bar{a}_O = \bar{a}_t = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & -\omega\cos\lambda & \omega\sin\lambda \\ -\omega R\cos\lambda & 0 & 0 \end{vmatrix} = -\omega^2 R \sin\lambda \cos\lambda \bar{j} - \omega^2 R \cos^2\lambda \bar{k} . \qquad (13.140)$$

The pseudo-force of transport is then (Fig. 13.16)

$$\bar{F}_t = -m\bar{a}_t = m\omega^2 R \sin\lambda \cos\lambda \bar{j} + m\omega^2 R \cos^2\lambda \bar{k} . \qquad (13.141)$$

The second projection is always positive and its effect is to reduce the local gravity force $m\bar{g}_\lambda = m(-g + \omega^2 R \cos^2\lambda)\bar{k}$ . Since the mass is a constant, the local gravity acceleration $g_\lambda$ on the surface of Earth is not a constant value in every point of the Earth's surface:

$$|g_\lambda| = g - \omega^2 R \cos^2\lambda . \qquad (13.142)$$

This formula provides the law of variation of $g_\lambda$ with respect to the latitude $\lambda$. The maxim of $g_\lambda$ is $g_{\lambda max}$=g, if $\lambda = \pm 90^\circ$ (at the two poles of the Earth) and the minimum of $g_\lambda$ is $g_{\lambda \min} = g - R\omega^2$ (at equator for $\lambda = 0^\circ$). The difference between the maximum and minimum values is
$R\omega^2 = 6376.5 \cdot 10^3 \cdot (72.722 \cdot 10^{-6})^2 = 0.0337\, m/s^2$ ,which is not always negligible.

The first projection of the pseudo-force of transport is a horizontal component pointing towards the equator. A heavy particle hanging by a string attached to a fixed point included in the local Oz axis, is supposed to indicate the direction towards the Earth's center, but the hanging string will not lie on this axis (Fig. 13.16).

The deviation angle ψ measured between the local radius direction (Oz) and the local weight vector $\overline{G}_\lambda$ is

$$\tan\psi = \frac{mR\omega^2 \sin\lambda\cos\lambda}{mg - mR\omega^2\cos^2\lambda} \simeq \frac{R\omega^2}{2g}\sin 2\lambda \,. \tag{13.143}$$

This angle has a maximum value for a latitude of 45°, for which $\tan\psi_{\max} = \dfrac{6376.5\cdot 10^3 \cdot \left(72.722\cdot 10^{-6}\right)^2}{2\cdot 9.807} = 0.001719 \Rightarrow \psi_{\max} \simeq 5'55''$

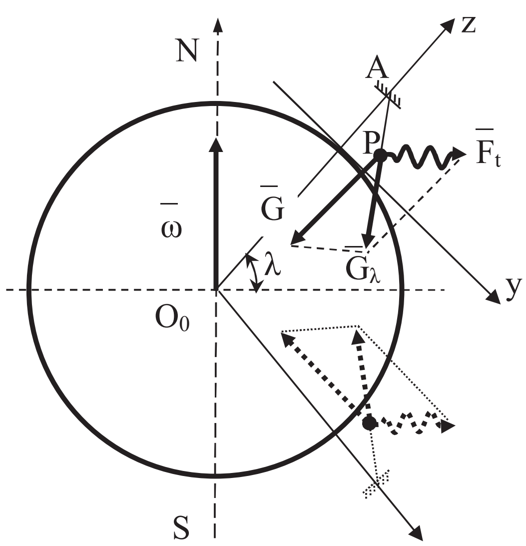



**Fig. 13.16 Earth's rotation influence on a hanging heavy particle**

### 13.10.2. Motion of a particle on Earth's surface

Two principal cases of motions will be considered: vertically falling particle and horizontally moving particle. In order to simplify the discussion, in both cases the instantaneous value of the velocity is *v*.

a) If a particle moves at a given moment with a velocity relative to Earth's surface $\overline{v}_r = -v\overline{k}$, then a Coriolis acceleration can be determined:

$$\overline{a}_c = 2\overline{\omega}\times\overline{v}_r = 2\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 0 & -\omega\cos\lambda & \omega\sin\lambda\\ 0 & 0 & -v\end{vmatrix} = 2\omega v\cos\lambda\overline{i} \qquad (13.144)$$

The Coriolis pseudo-force of inertia is then

$$\overline{F}_c = -m\overline{a}_c = -2m\omega v\cos\lambda\overline{i} \qquad (13.145)$$

Since the axis Ox is orientated towards west, a heavy particle does not fall along the local vertical indicated by $\overline{G}_\lambda$, but is **deviated towards east**.

b) A heavy particle of mass *m* has an instantaneous relative velocity $\overline{v}_r = v\cos\alpha\overline{i} + v\sin\alpha\overline{j}$ in the local horizontal plane. The angle α is arbitrary.

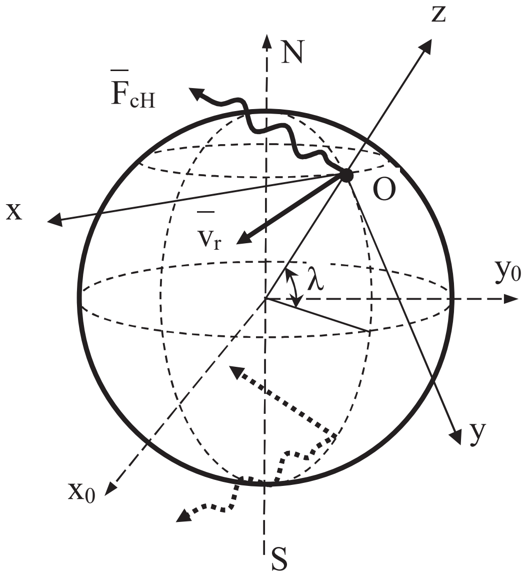



**Fig. 13.17 Motion deviation of a horizontally moving particle**

The Coriolis acceleration is in this case

$$\overline{a}_c = 2\overline{\omega}\times\overline{v}_r = 2\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 0 & -\omega\cos\lambda & \omega\sin\lambda\\ v\cos\alpha & v\sin\alpha & 0\end{vmatrix}$$
$$= 2\left(-\omega v\sin\alpha\sin\lambda\overline{i} + \omega v\cos\alpha\sin\lambda\overline{j} + \omega v\cos\alpha\cos\lambda\overline{k}\right) \qquad (13.146)$$

and the corresponding pseudo-force of inertia is :

$$\overline{F}_c = -m\overline{a}_c = 2m\omega v\left(\sin\alpha\sin\lambda\overline{i} - \cos\alpha\sin\lambda\overline{j} - \cos\alpha\cos\lambda\overline{k}\right) \qquad (13.147)$$

The most important effect of the Coriolis pseudo-force is in the plane Oxy, which is the vector containing only the first two components from (13.147) $\overline{F}_{cH} = 2m\omega v\left(\sin\alpha\overline{i} - \cos\alpha\overline{j}\right)\sin\lambda$. Being perpendicular by definition on the velocity vector, it can be easily proven its orientation to the "right" of the velocity if λ>0 and to the left in the other case. This effect is a motion **deviation to the right** in the northern hemisphere of the Earth and a deviation to the left in the southern hemisphere.

### 13.10.3. Foucault's Pendulum

Faucault's pendulum consists of a material point M of mass *m* attached at one end of a string of negligible weight and length *l*. The other end of the string is fixed to a point of coordinates A(0, 0, *l*) in the local frame presented in the above paragraphs. In these conditions, the material point is resting at equilibrium in the origin O, neglecting string deviation from the Oz axis.

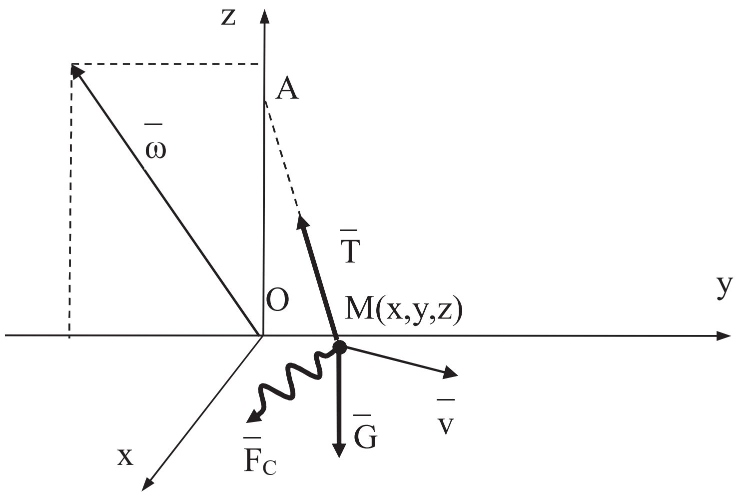



**Fig. 13.18 Forces on the pendulum of Foucault**

If T is the tension in the string, the force applied on the particle can be cast into vector form as $\overline{T} = T\frac{\overline{MA}}{\left|\overline{MA}\right|} = T\frac{-x\overline{i} - y\overline{j} + (l-z)\overline{k}}{l}$. Including the weight, the components of the applied forces are

$$X = \frac{-x}{l}T;\quad Y = \frac{-y}{l}T;\quad Z = \frac{(l-z)}{l}T - mg \qquad (13.148)$$

If the particle moves with a relative velocity $\overline{v}_r = \dot{x}\overline{i} + \dot{y}\overline{j} + \dot{z}\overline{k}$ the Coriolis acceleration becomes

$$\overline{a}_c = 2\overline{\omega}\times\overline{v}_r = 2\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 0 & -\omega\cos\lambda & \omega\sin\lambda\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix}$$
$$= 2\omega\left[-(\dot{z}\cos\lambda + \dot{y}\sin\lambda)\overline{i} + \dot{x}\sin\lambda\overline{j} + \dot{x}\cos\lambda\overline{k}\right] \qquad (13.149)$$

The Coriolis pseudo-force of inertia is :

$$\overline{F}_c = -m\overline{a}_c = 2m\omega\left[\left(\dot{z}\cos\lambda + \dot{y}\sin\lambda\right)\overline{i} - \dot{x}\sin\lambda\overline{j} - \dot{x}\cos\lambda\overline{k}\right]. \quad (13.150)$$

The differential equations of the relative motion are:

$$\begin{aligned} m\ddot{x} &= -T\frac{x}{l} + 2m\omega\left(\dot{z}\cos\lambda + \dot{y}\sin\lambda\right) \\ m\ddot{y} &= -T\frac{y}{l} - 2m\omega\dot{x}\sin\lambda \\ m\ddot{z} &= \frac{\left(l-z\right)}{l}T - mg - 2m\omega\dot{x}\cos\lambda \end{aligned} \quad (13.151)$$

If the motion of the particle is close to the origin, its vertical motion can be neglected, so that $z \simeq 0;\ \dot{z} \simeq 0$. Moreover, the tension in the string will be assumed to be a constant value $T = mg$. The first two equations form (13.151) are simplified

$$\begin{aligned} \ddot{x} - 2\omega\dot{y}\sin\lambda + g\frac{x}{l} &= 0 \\ \ddot{y} + 2\omega\dot{x}\sin\lambda + g\frac{y}{l} &= 0 \end{aligned} \quad (13.152)$$

Introducing $z = x + iy$ with $i = \sqrt{-1}$, the sum of the first equation with the second one multiplied by *i*, yields

$$\ddot{z} + 2\left(i\omega\sin\lambda\right)\dot{z} + p^2 z = 0. \quad (13.153)$$

with the notation $p^2 = \frac{g}{l}$.

The homogeneous differential equation (13.153) has the characteristic equation $\gamma^2 + 2\left(i\omega\sin\lambda\right)\gamma + p^2 = 0$ and the complex roots

$$\gamma_{1,2} = -i\omega\sin\lambda \pm i\sqrt{\omega^2\sin^2\lambda + p^2}\ .$$

The angular velocity of the Earth is small and its square can be neglected if compared to $p^2$. In this way the general solution of (13.153) can be expressed as

$$z = e^{-i\omega t\sin\lambda}\left(C_1 e^{ipt} + C_2 e^{-ipt}\right). \quad (13.154)$$

The sum of the complex numbers in the parenthesis $C_1 e^{ipt} + C_2 e^{-ipt}$, has an interesting graphic representation (Fig. 13.19). A **phasor** is a rotating vector of modulus $C_1$ and angle *pt* with the Ox axis. Similarly another one of modulus $C_2$ makes an angle (**phase**) *–pt* with the same axis.

Their scalars are constant, but the angle between them uniformly increases in time. Their sum can be obtained according to vector summation rule and a point is thus found on an ellipse of major axis $C_1+C_2$ and minor axis $C_1-C_2$. This ellipse is the general solution corresponding to $C_1e^{ipt}+C_2e^{-ipt}$.

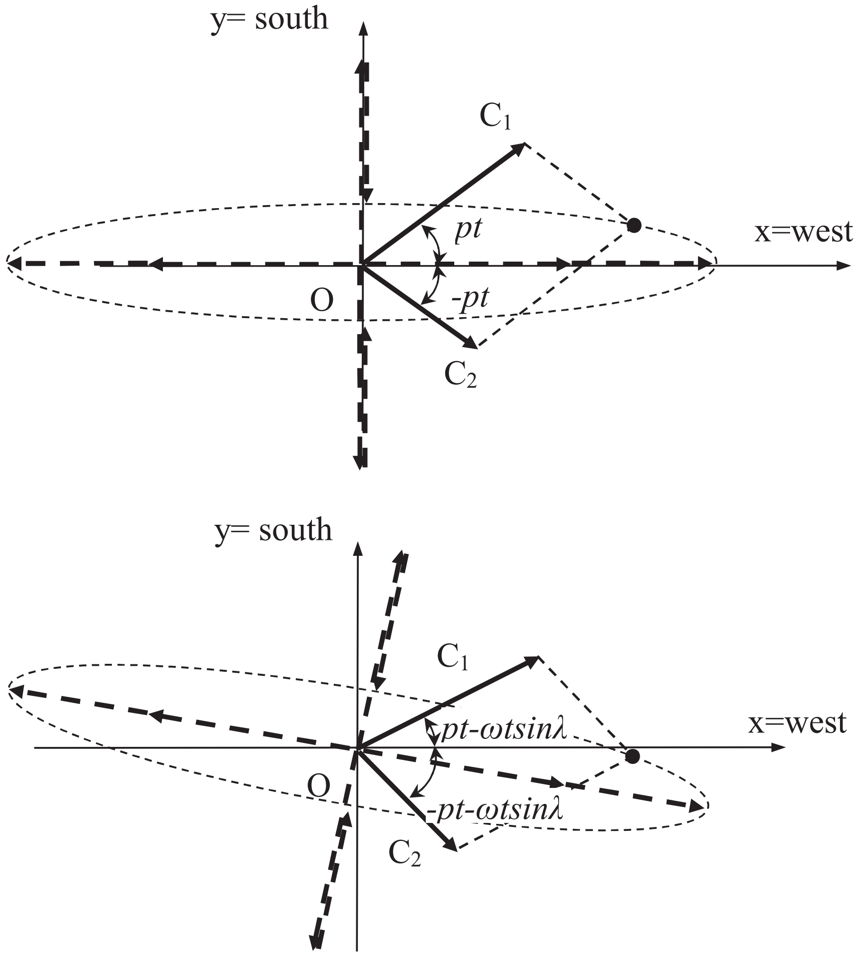


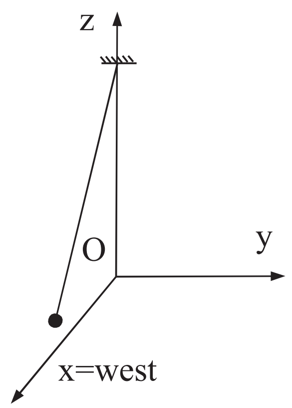

**Fig. 13.19 Elliptic motion in the Oxy plane in the absence of Earth's rotation**

Now, including the factor $e^{-i\omega t\sin\lambda}$, the phasors have the same scalars, but their phases are shifted by the same monotonic increasing angle $\omega t\sin\lambda$.
The general solution can be interpreted as a motion on an instantaneous elliptic motion, but on a uniformly rotating ellipse.

The period of this rotation at a latitude λ=45° is:

$$T=\frac{2\pi}{\omega\sin\lambda}=\frac{2\pi}{72.722\cdot10^{-6}\sin45^{\circ}}\simeq122188s=33.94hours$$

**Example.**




A pendulum of length *l=200m* is left to swing freely from a position which is at west from the attachment point. The initial angle is θ=10° between the string and the local vertical direction (Fig. 13.20).

Determine and plot the path of the material point attached, as a projection on the horizontal Oxy plane (Ox pointing west).

**Fig. 13.20 A pendulum at Earth's surface**

From eq. (13.154): $z = e^{-i\omega t \sin\lambda}\left(C_1 e^{ipt} + C_2 e^{-ipt}\right)$ it can be deduced the projections of the motion on the two axes:

$$x = \mathrm{Re}(z) = C_1 \cos(p - \omega \sin\lambda)t + C_2 \cos(p + \omega\sin\lambda)t$$

$$y = \mathrm{Im}(z) = C_1 \sin(p - \omega \sin\lambda)t - C_2 \sin(p + \omega\sin\lambda)t$$

$$\dot{x} = -C_1(p - \omega\sin\lambda)\sin(p - \omega\sin\lambda)t - C_2(p + \omega\sin\lambda)\sin(p + \omega\sin\lambda)t$$

$$\dot{y} = C_1(p - \omega\sin\lambda)\cos(p - \omega\sin\lambda)t - C_2(p + \omega\sin\lambda)\cos(p + \omega\sin\lambda)t$$

The initial conditions can be expressed as:

$$t = 0 \Rightarrow \{x = l\sin\theta;\;\; \dot{x} = 0;\;\; y = 0;\;\; \dot{y} = 0\}$$

Using the numerical values, $p = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.807}{200}} = 0.22144\, rad/s$, or an oscillation period of $T = \frac{2\pi}{p} = 28.374s$.

It follows $\begin{cases} C_1 + C_2 = l\sin\theta \simeq 34.73m \\ \frac{C_1}{C_2} = \frac{p + \omega\sin\lambda}{p - \omega\sin\lambda} = \frac{0.22145 + 72.722\cdot 10^{-6}\sin 45^\circ}{0.22145 - 72.722\cdot 10^{-6}\sin 45^\circ} \simeq 1.000464 \end{cases}$

With sufficient approximation $C_1 = C_2 \simeq 17.365m$

Consequently

$$x = 17.365\cos 0.22139t + 17.365\cos 0.22149t$$

$$y = 17.365\sin 0.22139t - 17.365\sin 0.22149t$$

The result is difficult to plot. In fact the pendulum makes 4306 oscillations for a complete rotation of the oscillation plane. A simplified plot with 20010 points equally spaced in time during half a rotation period is presented in Fig. 13.21. The general pattern corresponds however to the much denser trace of the effective motion.

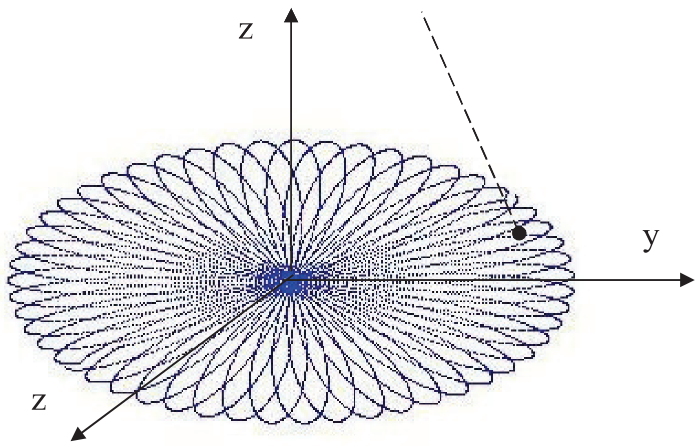



**Fig. 13.21 A subset of points on the path of a Foucault pendulum**

# 14. DYNAMICS OF A SYSTEM OF MATERIAL POINTS

## 14.1. Preliminaries

A system of material points $A_i$ of masses $m_i$ ($i = 1,...,n$) is considered. The fundamental equation of dynamics for all the material points belonging to this system forms the following system of differential equations:

$$m_i \overline{a}_i = \overline{F}_i + \sum_{j=1}^{n} \overline{F}_{ij} , \tag{14.1}$$

where $\overline{F}_i$ are external forces (Active and reaction forces originating from bodies and material points not belonging to the system) and $\overline{F}_{ij}$ are the internal forces (Forces of interaction between points $i$ and $j$ of the considered system, using the logical convention $\overline{F}_{ii} = \overline{0}$ as a particle cannot exert a force on itself). Internal forces occur in pairs and obey to the law of action and reaction. Consequently, the sum and the total moment of a pair of internal forces $\overline{F}_{ij}$ and $\overline{F}_{ji}$ are zero

$$\begin{aligned} &\overline{F}_{ij} + \overline{F}_{ji} = \overline{0} \\ &\overline{r}_i \times \left( \overline{F}_{ij} + \overline{F}_{ji} \right) = \overline{r}_i \times \overline{F}_{ij} + \overline{r}_i \times \overline{F}_{ji} = \overline{r}_i \times \overline{F}_{ij} + \overline{r}_j \times \overline{F}_{ji} = \overline{0} \end{aligned} \tag{14.2}$$

It follows that the resultant vector and the resultant moment of internal forces are also zero:

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{F}_{ij} = \overline{0} \\ &\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{r}_i \times \overline{F}_{ij} = \overline{0} \end{aligned} \tag{14.3}$$

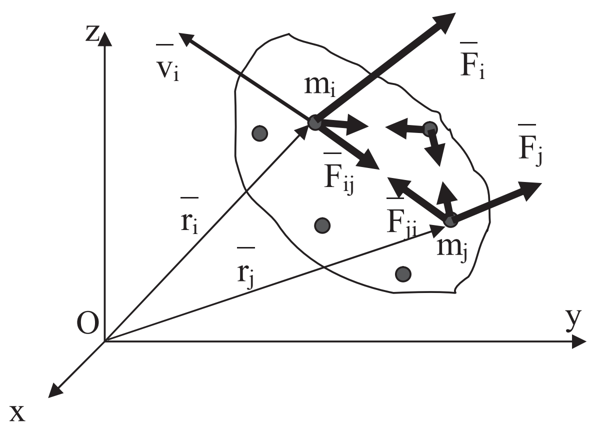



**Fig. 14.1 Forces on two material points from a system of material points**

Dynamic quantities such as linear momentum, angular momentum, kinetic energy, defined for a material point, can also be defined by a natural generalization to a system of material points.

The linear momentum of a material point of mass *m*, moving with a velocity $\bar{v}$ was defined as $\bar{H} = m\bar{v}$. It is natural to define the linear momentum of a system of material points:

$$\bar{H} = \sum_{i=1}^{n} m_i \bar{v}_i \tag{14.4}$$

The angular momentum of a material point about a fixed point O was defined by $\bar{K}_0 = \bar{r} \times m\bar{v}$. It is natural to define the angular momentum for a system of material points by:

$$\bar{K}_0 = \sum_{i=1}^{n} \bar{r}_i \times m_i \bar{v}_i . \tag{14.5}$$

The kinetic energy of a material point was defined by $T = \frac{1}{2} m v^2$. It is natural to define the kinetic energy for a system of material points by:

$$T \stackrel{def}{=} \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 . \tag{14.6}$$

The elementary work of a force was defined by $dW = \bar{F} d\bar{r}$. The **external elementary work** $dW_{ext}$ and the **internal elementary work** $dW_{\text{int}}$ for the forces acting on a system of material points are consequently defined:

$$dW_{ext} \stackrel{def}{=} \sum_{i=1}^{n} \bar{F}_i d\bar{r}_i ; \quad dW_{int} \stackrel{def}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{F}_{ij} d\bar{r}_i . \tag{14.7}$$

The implicit convention $\bar{F}_{ii} d\bar{r}_i = 0$ has been used in the summation.
If the force field is a conservative one, then a force function *U* and a potential energy *V* can be defined for force acting on a material point:

$$\begin{aligned} \sum_{i=1}^{n} \bar{F}_i d\bar{r}_i &= dU_{ext} = -dV_{ext} \\ \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{F}_{ij} d\bar{r}_i &= dU_{\text{int}} = -dV_{\text{int}} \end{aligned} \tag{14.8}$$

If forces of different natures are applied on the system then there are at most n force functions or potential energy expressions:

$$\begin{aligned} U_{ext} &= \sum U_i ; \quad U_{\text{int}} = \sum U_{ij} \\ V_{ext} &= \sum V_i ; \quad V_{\text{int}} = \sum V_{ij} \end{aligned} \tag{14.9}$$

in which $U_i$ and $V_i$ are respectively the force function and the potential energy corresponding to a certain external force $\overline{F}_i$ and $U_{ij}$, $V_{ij}$, are the force function and potential energy corresponding to a certain pair of internal force $\overline{F}_{ij}$ and $\overline{F}_{ji}$.

### *14.2. Theorem of linear momentum. Motion of the mass center*

The derivative with respect to time $t$ of the linear momentum $\overline{H}$ of a system of material points is equal to the resultant force vector of the external forces:

$$\frac{d\overline{H}}{dt} = \sum_{i=1}^{n} \overline{F}_i \,. \tag{14.10}$$

Proof.
Adding the differential equations (14.1), for all the system of particles it results

$$\sum_{i=1}^{n} m_i \overline{a}_i = \sum_{i=1}^{n} \overline{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} = \sum_{i=1}^{n} \overline{F}_i \,. \tag{14.11}$$

It has been used (14.3) to cancel the double sum of internal forces. The left side expression can be written successively:

$$\sum_{i=1}^{n} m_i \overline{a}_i = \sum_{i=1}^{n} m_i \frac{d\overline{v}_i}{dt} = \sum_{i=1}^{n} \frac{dm_i \overline{v}_i}{dt} = \frac{d}{dt}\sum_{i=1}^{n} m_i \overline{v}_i = \frac{d\overline{H}}{dt}, \tag{14.12}$$

and the theorem is thus proved.

<u>Theorem of the mass center motion</u>
The linear momentum $\overline{H} = \sum m_i \overline{v}_i$ can be written successively:

$$\overline{H} = \sum_{i=1}^{n} m_i \overline{v}_i = \sum_{i=1}^{n} m_i \frac{d\overline{r}_i}{dt} = \sum_{i=1}^{n} \frac{d\left(m_i \overline{r}_i\right)}{dt} = \frac{d}{dt}\sum_{i=1}^{n} m_i \overline{r}_i = \frac{d}{dt}\left(M\overline{\rho}\right)$$
$$= M\frac{d\overline{\rho}}{dt} = M\overline{v}_c . \tag{14.13}$$

By $M = \sum_{i=1}^{n} m_i$ is the mass of the system, $\overline{\rho}$ is the position vector of the mass center of the system and $\overline{v}_c$ is the velocity of the mass center C of the system of material points. If the momentum $\overline{H}$ is replaced by its expression (14.13) in relation (14.10) it follows that:

$$M\overline{a}_c = \sum_{i=1}^{n} \overline{F}_i \,. \tag{14.14}$$

The center of mass of a system of material points has the same motion (path, velocity, acceleration) as a material point having its mass equal to the mass of the

system and subjected to the action of a force equal with the resultant vector of the external forces.
Consequently, if the resultant vector of external forces is null (in particular, if the external forces are all null: $\overline{F_i} = 0$), then the linear momentum $\overline{H}$ is a constant. This theorem is called the conservation law of the linear momentum.

In these conditions, the mass center of a system of material points is at rest or has a uniform rectilinear motion.
Another interesting aspect is if the resultant vector of external forces is not null, and internal forces are considered, then the mass center of the system of material points, continues the initial motion, even if the particles change their individual motion.
For example, if a shell is exploding in the air its splinters shall have such motions that the center of mass of the system of splinters continues its parabolic path (if the drag force of the air is negligible).

Example. Rebound of a gun.
The system formed by the gun and the bullet is initially at rest. Only internal forces intervene during the firing process. If *M* is the mass of the gun, *m* is the mass of the bullet and *u* is the relative velocity of the bullet relative to the gun, it is to be determined the rebound velocity of the gun.

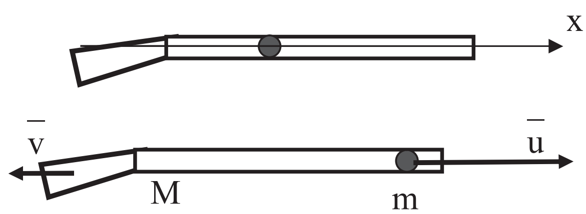



**Fig. 14.2 Rebound of a rifle or a gun**

If $\overline{v}$ is the rebound velocity of the gun assumed to be in the positive Ox direction, the absolute velocity of the bullet is $\overline{v} + \overline{u}$. Since the gun and the bullet are initially at rest, the linear momentum *H = 0*. Because no external forces intervene, the linear momentum is constant. It follows that:

$$Mv + m\left(v + u\right) = 0 \quad \Rightarrow \quad v = -\frac{mv}{M + m}u. \tag{14.15}$$

The minus sign corresponds to the well-known phenomenon of rebound of a gun which corresponds to the vector drawn in Fig. 14.2.

## 14.3. Theorem of angular momentum. Koenig's theorem. The theorem of angular momentum for a translating central frame

### 14.3.1. Theorem of angular momentum

The derivative with respect to time of the angular momentum $\overline{K}_O$ of a system of material points about a fixed point O is equal to the resultant moment of external forces about a same fixed point O:

$$\frac{d\overline{K}_0}{dt}=\sum_{i=1}^{n}\overline{r}_i\times\overline{F}_i\ . \tag{14.16}$$

Proof

Multiplying the differential equations (14.1) by $\overline{r}_i$ and adding these equations, it follows

$$\sum_{i=1}^{n}\overline{r}_i\times m_i\overline{a}_i=\sum_{i=1}^{n}\overline{r}_i\times\overline{F}_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{r}_i\times\overline{F}_{ij}\ . \tag{14.17}$$

By virtue of (14.3), $\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{r}_i\times\overline{F}_{ij}=\overline{0}$. The left side can be written successively:

$$\begin{aligned}\sum_{i=1}^{n}\overline{r}_i\times m_i\overline{a}_i&=\sum_{i=1}^{n}\overline{r}_i\times m_i\frac{d\overline{v}_i}{dt}=\sum_{i=1}^{n}\frac{d}{dt}\left(\overline{r}_i\times m_i\overline{v}_i\right)-\sum_{i=1}^{n}\frac{d\overline{r}_i}{dt}\times m_i\overline{v}_i=\\&=\frac{d}{dt}\left(\sum_{i=1}^{n}\overline{r}_i\times m_i\overline{v}_i\right)=\frac{d\overline{K}_0}{dt}\end{aligned} \tag{14.18}$$

because $\frac{d\overline{r}_i}{dt}=\overline{v}_i$ and obviously $\overline{v}_i\times m_i\overline{v}_i=0$. The theorem is thus proofed.

### 14.3.2. Koenig's theorem for the angular momentum

Two frames are considered:

- a fixed Cartesian frame Oxyz and
- a moving Cartesian frame $Cx'y'z'$ of origin C (the mass center of the system of material points) with axes remaining parallel to the axes of the fixed frame. This is called a **central frame**.

The position vector of a certain material point $m_i$ about the fixed and respectively the moving frame are $\overline{r}_i$, $\overline{r}_i'$. $\overline{\rho}$ is the position vector of the mass center C about fixed frame. It follows that:

$$\overline{r}_i=\overline{\rho}+\overline{r}_i'\ . \tag{14.19}$$

Differentiating this relation with respect to the time, it can be obtained:

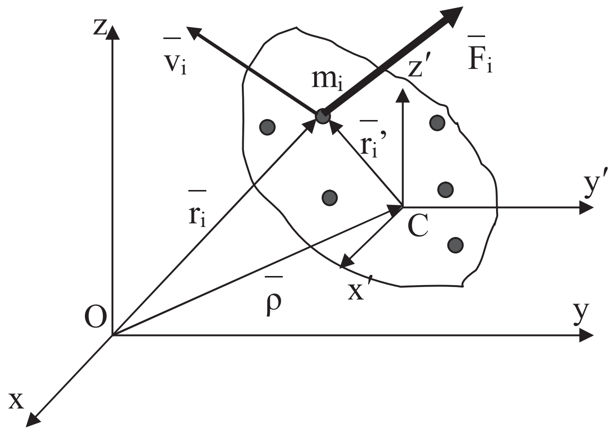



**Fig. 14.3 System of material points in a moving central frame**

$$\overline{v}_i = \overline{v}_C + \overline{v}_i{}' . \tag{14.20}$$

The angular momentum about the fixed point O can be written successively:

$$\begin{aligned}\overline{K}_O &= \sum_{i=1}^{n} \overline{r}_i \times m_i \overline{v}_i = \sum_{i=1}^{n} \left(\overline{\rho} + \overline{r}_i'\right) \times m_i \left(\overline{v}_C + \overline{v}_i'\right) \\ &= \sum_{i=1}^{n} \overline{\rho} \times m_i \overline{v}_C + \sum_{i=1}^{n} \overline{\rho} \times m_i \overline{v}_i' + \sum_{i=1}^{n} \overline{r}_i' \times m_i \overline{v}_C + \sum_{i=1}^{n} \overline{r}_i' \times m_i \overline{v}_i' \\ &= \overline{\rho} \times \left(\sum_{i=1}^{n} m_i\right) \overline{v}_C + \overline{\rho} \times \sum_{i=1}^{n} m_i \overline{v}_i' + \left(\sum_{i=1}^{n} m_i \overline{r}_i'\right) \times \overline{v}_C + \sum_{i=1}^{n} \overline{r}_i' \times m_i \overline{v}_i'\end{aligned} \tag{14.21}$$

from which

$$\overline{K}_O = \overline{\rho} \times M \overline{v}_C + \overline{K}_C . \tag{14.22}$$

The relation is known as **Koenig’s theorem for the angular momentum**: the angular momentum of a system of material points about a fixed point O is equal to the sum of the angular momentum of a single material point placed at the center of mass but having a mass equal to the whole mass of the system, and the angular momentum of the system of material points in their relative motion about the center of mass.

In the proof use was made of the facts: $\sum_{i=1}^{n} m_i \overline{r}_i' = 0$ (The static moment about the mass center is null. See Chapter 3), $\sum m_i \overline{v}_i{}' = \frac{d}{dt}\left(\sum m_i \overline{r}_i{}'\right) = 0$ since there is no frame rotation.

### 14.3.3. The theorem of angular momentum in a translating central frame

The theorem of angular momentum has the same form for the relative motion of a system of material points about its translating central frame.

Proof

Replacing $\bar{K}_O$ by its expression ((14.22) Koenig theorem) and the position vector by its expression (14.19) in the angular momentum theorem (14.16), the following expression is obtained

$$\frac{d}{dt}\left(\bar{\rho}\times M\bar{v}_C+\bar{K}_C\right)=\sum_{i=1}^{n}\left(\bar{\rho}+\bar{r}_i'\right)\times\bar{F}_i. \quad (14.23)$$

Developing the vector calculus

$$\frac{d\bar{\rho}}{dt}\times M\bar{v}_C+\bar{\rho}\times M\frac{d\bar{v}_C}{dt}+\frac{d\bar{K}_C}{dt}=\bar{\rho}\times\sum_{i=1}^{n}\bar{F}_i+\sum_{i=1}^{n}\bar{r}_i'\times\bar{F}_i. \quad (14.24)$$

Since $\frac{d\bar{\rho}}{dt}\times M\bar{v}_C=\bar{v}_C\times M\bar{v}_C=0$ and $M\frac{d\bar{v}_C}{dt}=M\bar{a}_C=\sum_{i=1}^{n}\bar{F}_i$ (theorem of linear momentum using the mass center). After reduction of terms, the theorem becomes

$$\frac{d\bar{K}_C}{dt}=\sum_{i=1}^{n}\bar{r}_i'\times\bar{F}_i. \quad (14.25)$$

The validity of the theorem of the angular momentum for the relative motion of a system of material points about its center of mass is thus proved.

Example

Two material points of masses *m* rotate in the plane *Cy'z'* around the *Cx'* axis by the angle $\varphi(t)$ and their distances to the *Cx'* axis are both *l*. The mass center C moves with constant velocity $\bar{v}_C$ along the *Cx'* axis. The *Cx'y'z'* frame remains with axes parallel to those of the fixed frame *Oxyz* during motion.

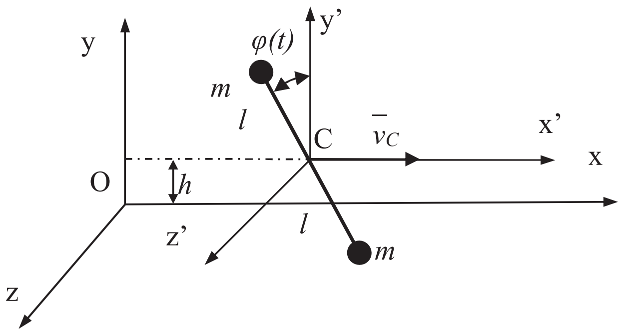



**Fig. 14.4 Two material points moving in a translating frame**

. Assuming only gravitational forces, write the angular momentum about O and apply the theorem of angular momentum for the system of two material points.

The coordinates of the two material points are $\left(x_1' = 0;\ y_1' = l\cos\varphi;\ z_1' = l\sin\varphi\right)$ and $\left(x_2' = 0;\ y_2' = -l\cos\varphi;\ z_2' = -l\sin\varphi\right)$ velocities of the two material points relative to the central frame are:

$$\bar{v}_1' = \begin{vmatrix} \bar{i}' & \bar{j}' & \bar{k}' \\ \dot{\varphi} & 0 & 0 \\ x_1' & y_1' & z_1' \end{vmatrix} = l\dot{\varphi}\left(-\sin\varphi\bar{j}' + \cos\varphi\bar{k}'\right);\quad \bar{v}_2' = \begin{vmatrix} \bar{i}' & \bar{j}' & \bar{k}' \\ \dot{\varphi} & 0 & 0 \\ x_2' & y_2' & z_2' \end{vmatrix} = l\dot{\varphi}\left(\sin\varphi\bar{j}' - \cos\varphi\bar{k}'\right).$$

According to (14.22): $\bar{\rho} \times M\bar{v}_C = \left(x_C\bar{i} + h\bar{j}\right) \times 2mv_C\bar{i} = -2mhv_C\bar{k}$ and

$$\bar{K}_C = ml\dot{\varphi}\begin{vmatrix} \bar{i}' & \bar{j}' & \bar{k}' \\ 0 & l\cos\varphi & l\sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{vmatrix} + ml\dot{\varphi}\begin{vmatrix} \bar{i}' & \bar{j}' & \bar{k}' \\ 0 & -l\cos\varphi & -l\cos\varphi \\ 0 & \sin\varphi & -\cos\varphi \end{vmatrix} = 2ml^2\dot{\varphi}\bar{i}'$$

The moments of the two weight forces are:

$$\bar{M}_C\left(\bar{G}_1\right) = \begin{vmatrix} \bar{i}' & \bar{j}' & \bar{k}' \\ x_1' & y_1' & z_1' \\ 0 & 0 & mg \end{vmatrix} = mgl\cos\varphi\bar{i}';\quad \bar{M}_C\left(\bar{G}_2\right) = \begin{vmatrix} \bar{i}' & \bar{j}' & \bar{k}' \\ x_2' & y_2' & z_2' \\ 0 & 0 & mg \end{vmatrix} = -mgl\cos\varphi\bar{i}'$$

Using the theorem of angular momentum (14.25) it follows:

$$\frac{d\bar{K}_C}{dt} = 2ml^2\ddot{\varphi}\bar{i}' = mgl\cos\varphi\bar{i}' - mgl\cos\varphi\bar{i}' = \bar{0}\text{, so that } \ddot{\varphi} = 0$$

which proves that this helical motion takes place with constant angular velocity.

## *14.4. Theorem of kinetic energy and work. Koenig's theorem. The validity of the theorem of kinetic energy and work in a translating central frame*

### 14.4.1. Theorem of kinetic energy and work

The differential of the kinetic energy of a system of material points is equal to the sum of the elementary external and internal work:

$$dT = dW_{ext} + dW_{\text{int}} \tag{14.26}$$

Proof

Multiplying the differential equations (14.1) by $d\bar{r}_i$ and afterwards adding these equations, one gets

$$\sum_{i=1}^{n} m_i\bar{a}_i \cdot d\bar{r}_i = \sum_{i=1}^{n}\bar{F}_i d\bar{r}_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\bar{F}_{ij}d\bar{r}_i\,. \tag{14.27}$$

According to the definitions of external and respectively internal work are $dW_{ext} = \sum_{i=1}^{n}\bar{F}_i d\bar{r}_i$; $dW_{\text{int}} = \sum_{i=1}^{n}\sum_{j=1}^{n}\bar{F}_{ij}d\bar{r}_i$ and for the left side of the theorem

$$\sum m_i \bar{a}_i \cdot d\bar{r}_i = \sum m_i \frac{d\bar{v}_i}{dt} \cdot d\bar{r}_i = \sum m_i \bar{v}_i \cdot d\bar{v}_i = \sum m_i d\left(\frac{\bar{v}_i^2}{2}\right) = d\left(\sum \frac{1}{2} m_i v_i^2\right) = dT \quad (14.28)$$

The theorem is thus proved. It can be remarked that this theorem does not eliminate the internal forces because in general $dW_{\text{int}} \neq 0$. However, there are some cases when $dW_{\text{int}} = 0$. Considering only two material points from the system, the expression:

$$\bar{F}_{ij} d\bar{r}_i + \bar{F}_{ji} d\bar{r}_j = \bar{F}_{ij} d\bar{r}_i - \bar{F}_{ij} d\bar{r}_j = \bar{F}_{ij}\left(\bar{v}_i - \bar{v}_j\right) dt = \bar{F}_{ij} \bar{v}_{ij} dt . \quad (14.29)$$

This expression is zero in one of the following cases:

a) The system of material points is a **rigid system** or rigid body, because $\bar{F}_{ij} \perp \bar{v}_{ij}$.
   Explanation comes from the constant distance between points: $\left|\overline{M_i M_j}\right| = \text{const.}$ and consequently one point $M_j$ moves on a sphere centered at $M_i$.
b) There are only **smooth constraints**, because $\bar{F}_{ij} \perp \bar{v}_{ij}$.
   For a smooth surface constraining the system, the relative velocity $\bar{v}_{ij}$ is situated in the tangent plane and $\bar{F}_{ij}$ is normal to the surface of contact. If the constraint is a hinge, which can be modeled as a fixed point of a rigid body, hence the velocity $\bar{v}_{ij} = 0$ in the application point of the reaction forces.
c) There are rough constraints and the relative motion is a **rolling motion** (without sliding).
   The explanation comes from $\bar{v}_{ij} = 0$ (the point of contact is an instantaneous center of rotation).
d) The system has two points connected by **inextensible strings**.
   In this case $\bar{F}_{ij} \perp \bar{v}_{ij}$ if the string is stretched (as in case of a rigid system) and $\bar{F}_{ij} = 0$ if the string is loosed.

### 14.4.2. Koenig's theorem for kinetic energy and work

The same mechanical system and frames are considered, as indicated in Fig. 14.3 and the relations (14.20) and (14.26) are considered. The kinetic energy about the fixed frame Oxyz can be written successively:

$$\begin{aligned} T &= \sum_{i=1}^{n} \frac{1}{2} m_i \bar{v}_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \left(\bar{v}_C + \bar{v}_i'\right)^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \bar{v}_C^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \bar{v}'^2 + \sum_{i=1}^{n} m_i \bar{v}_C \bar{v}' \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right) \bar{v}_C^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \bar{v}'^2 + \bar{v}_C \sum_{i=1}^{n} m_i \bar{v}' \end{aligned} \quad (14.30)$$

The last term is null as the derivative of a static moment determined about the mass center, so that the theorem becomes

$$T = \frac{1}{2} M v_C^2 + T_C . \quad (14.31)$$

This formula is known as Koenig's theorem for the kinetic energy: "The kinetic energy of a system of material points in a fixed frame equals the sum of the kinetic energy of a single material point placed at the mass center but having a mass equal to the mass of the system and the kinetic energy of the system in its relative motion about the mass center".
The elementary external work can be written

$$dW_{ext} = \sum_{i=1}^{n} \overline{F}_i d\overline{r}_i = \sum_{i=1}^{n} \overline{F}_i d\left(\overline{\rho} + \overline{r}_i{}'\right) = d\overline{\rho}\sum_{i=1}^{n} \overline{F}_i + \sum_{i=1}^{n} \overline{F}_i d\overline{r}_i{}'$$
$$= d\overline{\rho}\sum_{i=1}^{n} \overline{F}_i + dW_{extC} \tag{14.32}$$

which represents a sum of two elementary works: one of the applied forces with the elementary displacement of the mass center and the other is an elementary work of applied forces with the relative displacements of points about the translating central system. The internal elementary work :

$$dW_{int} \overset{def}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} d\overline{r}_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} d\left(\overline{\rho} + \overline{r}_i'\right) = d\overline{\rho}\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} d\overline{r}_i'$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{F}_{ij} d\overline{r}_i' = dW_{intC} \tag{14.33}$$

is the same in the fixed and the translating central frame.

### 14.4.3. Theorem of energy and work in a translating central frame

This theorem states that the theorem of kinetic energy and work has the same form for the relative motion of a system of material points about a translating central frame:

$$dT_C = dW_{extC} + dW_{\text{int}C}\,. \tag{14.34}$$

Proof

Replacing the kinetic energy T by its expression (14.31) and $\overline{r}_i$ by its expression in the translating frame, the position vector (14.19) in the proven theorem (14.26), one gets

$$d\left(\frac{1}{2}Mv_C^2 + T_C\right) = d\overline{\rho}\sum_{i=1}^{n} \overline{F}_i + dW_{extC} + dW_{\text{int}C}\,. \tag{14.35}$$

Since: $d\left(\frac{1}{2}Mv_C^2\right) = M\overline{v}_C d\overline{v}_C = M\frac{d\overline{\rho}}{dt}d\overline{v}_C = M\frac{d\overline{\rho}}{dt}\overline{a}_C dt = M\overline{a}_C d\overline{\rho} = \left(\sum_{i=1}^{n} \overline{F}_i\right)d\overline{\rho}$,

the last expression simplifies to

$$d\left(T_C\right) = dW_{extC} + dW_{\text{int}C}\,. \tag{14.36}$$

which proves the theorem.

# 15. DYNAMICS OF A RIGID BODY

## 15.1. Dynamic quantities for a rigid body

Since a rigid body is continuous, the sums appearing in the previous chapter, concerning the definitions of the linear momentum, the angular momentum and the kinetic energy must be replaced by integrals. The following are the expressions for linear, angular momentum and kinetic energy for a rigid body:

$$\bar{H} = \int_D \bar{v}dm = M\bar{v}_c; \quad \bar{K}_O = \int_D \bar{r}\times\bar{v}dm; \quad T = \frac{1}{2}\int_D v^2 dm \tag{15.1}$$

The integrals are defined on the three dimensional domain ($D$) occupied by the rigid body, $M$ is its mass and $\bar{v}_c$ is the velocity of its mass center. These expressions have specific forms for particular motions of a rigid body:

a) <u>Translation:</u>

$$\begin{aligned}
&\bar{H} = M\bar{v}_c;\\
&\bar{K}_o = \int_D \bar{r}\times\bar{v}_c dm = \left(\int_D \bar{r}dm\right)\times\bar{v}_c = M\bar{\rho}\times\bar{v}_c = \bar{\rho}\times M\bar{v}_c\\
&T = \int_D \frac{1}{2}v_c^2 dm = \frac{1}{2}v_c^2\int_D dm = \frac{1}{2}Mv_c^2
\end{aligned} \tag{15.2}$$

It follows that the linear momentum, the angular momentum and the kinetic energy are equal to those of a material point placed at the center of mass and having a mass equal to the mass of the rigid body ($M$). The usual notations have been used: $\bar{\rho}$; $\bar{v}_c$ are the mass center position vector and respectively velocity.

b) <u>Motion of a rigid body with a fixed point:</u>
The general formula for the linear momentum remains the same

$$\bar{H} = M\bar{v}_c \tag{15.3}$$

in which $\bar{v}_C = \bar{\omega}\times\bar{\rho}$.
The velocity for an arbitrary point of rigid body in this case of motion is $\bar{v} = \bar{\omega}\times\bar{r}$.
Consequently the angular momentum can be written:

$$\begin{aligned}
\bar{K}_0 &= \int_D \bar{r}\times\left(\bar{\omega}\times\bar{r}\right)dm =\\
&= \int_D \left(x\bar{i} + y\bar{j} + z\bar{k}\right)\times\left(\left(\omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}\right)\times\left(x\bar{i} + y\bar{j} + z\bar{k}\right)\right)dm.
\end{aligned} \tag{15.4}$$

The double cross product can be developed using the formula $\bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$, yielding:

$$\begin{aligned}
\bar{K}_0 &= \int_D \left[ \left( x\bar{i} + y\bar{j} + z\bar{k} \right)^2 \left( \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k} \right) - \left( \omega_x x + \omega_y y + \omega_z z \right)\left( x\bar{i} + y\bar{j} + z\bar{k} \right) \right] dm \\
&= \int_D \left[ \left( x^2 + y^2 + z^2 \right)\left( \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k} \right) - \left( \omega_x x + \omega_y y + \omega_z z \right)\left( x\bar{i} + y\bar{j} + z\bar{k} \right) \right] dm \\
&= \bar{i} \int_D \left[ \left( y^2 + z^2 \right)\omega_x - xy\omega_y - xz\omega_z \right] dm + \bar{j} \int_D \left[ \left( z^2 + x^2 \right)\omega_y - yz\omega_z - yx\omega_x \right] dm \\
&+ \bar{k} \int_D \left[ \left( x^2 + y^2 \right)\omega_z - zx\omega_x - zy\omega_y \right] dm.
\end{aligned} \tag{15.5}$$

By identification of the moments and products of inertia expressions, the last formula can be shorter written:

$$\begin{aligned}
\bar{K}_0 = &\left( J_x \omega_x - J_{xy}\omega_y - J_{xz}\omega_z \right)\bar{i} \\
&+ \left( -J_{yx}\omega_x + J_y \omega_y - J_{yz}\omega_z \right)\bar{j}\ , \\
&+ \left( -J_{zx}\omega_x - J_{zy}\omega_y + J_z \omega_z \right)\bar{k}
\end{aligned} \tag{15.6}$$

in which $J_x$, $J_y$, $J_z$ are the moments of inertia and $J_{xy}$, $J_{yz}$ , $J_{zx}$ are the products of inertia of the rigid body with respect to the Ox, Oy and Oz axes (see chapter 4). The kinetic energy can be obtained in this case:

$$\begin{aligned}
T &= \int_D \frac{1}{2} \left( \bar{\omega} \times \bar{r} \right)^2 dm = \\
&= \frac{1}{2} \int_D \left[ \left( \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k} \right) \times \left( x\bar{i} + y\bar{j} + z\bar{k} \right) \right]^2 dm = \\
&= \frac{1}{2} \int_D \left[ \left( \omega_y z - \omega_z y \right)\bar{i} + \left( \omega_z x - \omega_x z \right)\bar{j} + \left( \omega_x y - \omega_y x \right)\bar{k} \right]^2 dm \\
&= \frac{1}{2} \int_D \begin{bmatrix} \omega_x^2 \left( y^2 + z^2 \right) + \omega_y^2 \left( z^2 + x^2 \right) + \omega_z^2 \left( x^2 + y^2 \right) \\ -2\omega_x \omega_y xy - 2\omega_y \omega_z yz - 2\omega_z \omega_x zx \end{bmatrix} dm
\end{aligned} \tag{15.7}$$

Again identifying the moments of inertia and products of inertia, the kinetic energy expression becomes:

$$T = \frac{1}{2}\left( J_x \omega_x^2 + J_y \omega_y^2 + J_z \omega_z^2 - 2J_{xy}\omega_x \omega_y - J_{yx}\omega_y \omega_z - J_{zx}\omega_z \omega_x \right). \tag{15.8}$$

The expressions of the angular momentum and kinetic energy can be written in a compact matrix form as:

$$\{K_0\} = [J_0]\{\omega\}, \qquad T = \frac{1}{2}\{\omega\}^T [J_0]\{\omega\}. \tag{15.9}$$

where $[J_O]$ is the square matrix of the moments of inertia (called **inertia matrix**) and $\{\omega\}$ and $\{K_O\}$ are the column matrices of angular velocity and angular momentum respectively:

$$[J_0] = \begin{bmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{bmatrix}; \qquad \{\omega\} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}; \qquad \{K_0\} = \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} \quad (15.10)$$

c) Rotation about an axis

The Oz axis is placed along the rotation axis. The angular velocity components are $\omega_x = 0;\ \omega_y = 0;\ \omega_z = \omega$.

The following formulas are obtained as particular case of the previous case

$$\begin{aligned} &\bar{H} = M\bar{v}_c; \\ &\bar{K}_0 = -J_{xz}\omega\bar{i} - J_{yz}\omega\bar{j} + J_z\omega\bar{k}\,. \\ &T = \frac{1}{2}J_z\omega^2 \end{aligned} \qquad (15.11)$$

d) The general motion of a rigid body

Applying Koenig's theorems it follows that:

$$\begin{aligned} &\bar{H} = M\bar{v}_c; \\ &\bar{K}_0 = \bar{\rho} \times M\bar{v}_c + \bar{K}_c\,, \\ &T = \frac{1}{2}Mv_c^2 + T_c \end{aligned} \qquad (15.12)$$

in which $\bar{K}_c$ and $T_c$ have the expressions (15.6) and (15.8) deduced for a rigid body with a fixed point. The origin of the translating frame is at the mass center C, because the relative motion of a rigid body about its mass center can be seen as a motion of a rigid body around a fixed point C.

## *15.2. Rigid body in rotation about a fixed axis*

A rigid body having two distinct fixed points (smooth spherical joints) $O_1$ and $O_2$ (fig. 15.1) acted by the given forces $F_i (i = 1,...,n)$ is rotating about the fixed axis $O_1O_2$. A fixed Cartesian frame $O_1x_1y_1z_1$ is considered, so that $O_1z_1$ is the rotation axis. Another movable Cartesian frame *Oxyz*, is assumed attached to the body, with *Oz* as the rotation axis ($O \equiv O_1$) and the *Oxz* - plane such that the mass center $C(\xi, 0, \zeta)$ of the rigid body is situated in it.

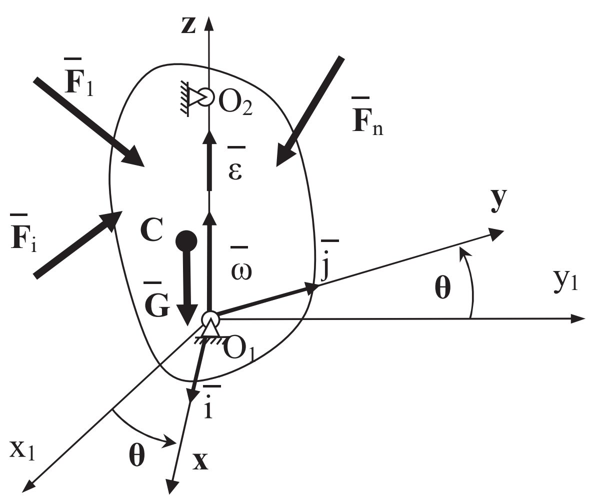



**Fig. 15.1 Rigid body in rotation about the Oz axis**

The distance between the fixed points is denoted $h = O_1O_2$, the rotation angle θ is measured between mobile *Ox* and fixed $Ox_1$ axes and *X, Y, Z, Mox, Moy, Moz* are the projections of the resultant force vector and respectively the resultant moment vector of the given applied forces $\overline{F_i}$ on the axes of the moving frame Oxyz. If the two spherical joints are replaced by the reactions $\overline{R}_1\left(R_{1x}, R_{1y}, R_{1z}\right)$ and $\overline{R}_2\left(R_{2x}, R_{2y}, R_{2z}\right)$, the velocity of the mass center is

$$\overline{v}_c = \overline{\omega} \times \overline{\rho} = \omega \overline{k} \times \left(\xi \overline{i} + \zeta \overline{k}\right) = \xi \omega \overline{j}\,, \qquad (15.13)$$

then the expressions of dynamic quantities become:

$$\begin{aligned} &\overline{H} = M\overline{v}_c = M\xi\omega\overline{j} \\ &\overline{K}_0 = -J_{xz}\omega\overline{i} - J_{yz}\omega\overline{j} + J_z\omega\overline{k} \end{aligned} \qquad (15.14)$$

The theorems of linear and angular momentum yield in this case:

$$\begin{aligned} &\frac{d\overline{H}}{dt} = \frac{\partial \overline{H}}{\partial t} + \overline{\omega} \times \overline{H} = M\xi\dot{\omega}\overline{j} + \omega\overline{k} \times M\xi\omega\overline{j} = -M\xi\omega^2\overline{i} + M\xi\varepsilon\overline{j} \\ &\qquad = \overline{R} + \overline{R}_1 + \overline{R}_2 \\ &\frac{d\overline{K}_0}{dt} = \frac{\partial \overline{K}_0}{\partial t} + \overline{\omega} \times \overline{K} = -J_{xz}\dot{\omega}\overline{i} - J_{yz}\dot{\omega}\overline{j} + J_z\dot{\omega}\overline{k} + \\ &+\omega\overline{k} \times \left(-J_{xz}\omega\overline{i} - J_{yz}\omega\overline{j} + J_z\omega\overline{k}\right) = \left(-J_{xz}\varepsilon + J_{yz}\omega^2\right)\overline{i} + \\ &+\left(-J_{yz}\varepsilon - J_{xz}\omega^2\right)\overline{j} + J_z\varepsilon\overline{k} = \overline{M}_0 + \overline{O_1O_2} \times \overline{R}_2 \end{aligned} \qquad (15.15)$$

Projecting these equations on the moving frame axes, it follows

$$\begin{aligned}
&-M\xi\omega^2 = X + R_{1x} + R_{2x}\\
&M\xi\varepsilon = Y + R_{1y} + R_{2y}\\
&0 = Z + R_{1z} + R_{2z}\\
&-J_{xz}\varepsilon + J_{yz}\omega^2 = M_{ox} - hR_{2y}\\
&-J_{yz}\varepsilon - J_{xz}\omega^2 = M_{oy} + hR_{2x}\\
&J_z\varepsilon = M_{oz}
\end{aligned} \tag{15.16}$$

The theorem of energy and work $dE = dW_{ext}$, since $dW_{int} = 0$ as the mechanical system is a rigid one, can be written:

$$\begin{aligned}
&dT = d\left(\frac{1}{2}J_z\omega^2\right) = J_z\omega d\omega;\\
&dW_{ext} = \sum_{i=1}^{n}\overline{F}_i d\overline{r}_i = \sum_{i=1}^{n}\overline{F}_i\overline{v}_i dt = \sum_{i=1}^{n}\overline{F}_i\left(\overline{\omega}\times\overline{r}_i\right)dt = \overline{\omega}\sum_{i=1}^{n}\left(\overline{r}_i\times\overline{F}_i\right)dt\\
&= \overline{\omega}\overline{k}\cdot\left(M_{ox}\overline{i} + M_{oy}\overline{j} + M_{oz}\overline{k}\right)dt = M_{oz}\omega dt = M_{oz}d\theta\\
&dT = dW_{ext} \Rightarrow J_z\omega d\omega = M_{oz}d\theta \Rightarrow J_z\omega\frac{d\omega}{dt} = M_{oz}\frac{d\theta}{dt}
\end{aligned} \tag{15.17}$$

But $\frac{d\omega}{dt} = \varepsilon, \frac{d\theta}{dt} = \omega$ and simplifying by $\omega \neq 0$, is found again the last relation from (15.16). The system of differential equations (15.16) completely determines the motion of the rigid body, by integrating the last equation:

$$J_z\varepsilon = M_{oz} \tag{15.18}$$

To be remarked the similarity of this differential equation to the fundamental equation of dynamics $M\overline{a} = \overline{F}$.
From the remaining five equations it is possible to determine the projections of the reaction forces $R_{1x}$, $R_{1y}$, $R_{2x}$, $R_{2y}$ :

$$\begin{aligned}
&R_{1x} = -M\xi\omega^2 - X + \left(J_{yz}\varepsilon + J_{xz}\omega^2 + M_{oy}\right)/h\\
&R_{1y} = M\xi\varepsilon - Y - \left(J_{xz}\varepsilon - J_{yz}\omega^2 + M_{ox}\right)/h\\
&R_{2x} = \left(-J_{yz}\varepsilon - J_{xz}\omega^2 M_{oy}\right)/h\\
&R_{2y} = \left(J_{xz}\varepsilon - J_{yz}\omega^2 + M_{ox}\right)/h
\end{aligned} \tag{15.19}$$

The projections on the Oz axis $R_{1z}$ and $R_{2z}$ are indeterminable, because there is a single equation (the third) with two unknowns. The conclusion is that two spherical joints are not necessary to immobilize the axis of rotation. For example, if $O_2$ would be a hinge and $O_1$ a spherical joint, the axis $O_1O_2$ remains fixed and $R_{2z}$ =0.
Therefore $R_{1z} = -Z; \quad R_{2z} = 0$.

Application

An important practical application is **rotor balance**. A rotor is a rigid body of axial symmetry about the Oz axis. The perfect symmetry cannot be technologically obtained, but the mechanical conditions for such case are required in this application.

The reactions of the rotor in the initial state of rest are $\overline{R}_1^S\left(R_{1x}^S, R_{1y}^S, R_{1z}^S\right)$ and $\overline{R}_2^S\left(R_{2x}^S, R_{2y}^S, R_{2z}^S\right)$. These static reactions are the solutions of the following system of equations, obtained from (15.16) for $\omega \equiv 0$:

$$\begin{aligned}
0 &= X - R_{1x}^S + R_{2x}^S \\
0 &= Y + R_{1y}^S + R_{2y}^S \\
0 &= Z + R_{1z}^S + R_{2z}^S \\
0 &= M_{ox} - hR_{2y}^S \\
0 &= M_{oy} + hR_{2x}^S
\end{aligned} \tag{15.20}$$

Subtracting the equations (15.20) from (15.16) and denoting by $\overline{R}_1^d = \overline{R}_1 - \overline{R}_1^S$ and $\overline{R}_2^d = \overline{R}_2 - \overline{R}_2^S$ the **dynamic reactions**, the equations of motion become:

$$\begin{aligned}
-M\xi\omega^2 &= R_{1x}^d + R_{2x}^d \\
M\xi\varepsilon &= R_{1y}^d + R_{2y}^d \\
0 &= R_{1z}^d + R_{2z}^d \ . \\
-J_{xz}\varepsilon + J_{yz}\omega^2 &= -hR_{2y}^d \\
-J_{yz}\varepsilon - J_{xz}\omega^2 &= hR_{2x}^d
\end{aligned} \tag{15.21}$$

If the angular velocity ω increases, these dynamic reactions increase very fast as they are proportional to ω². It is important to have null dynamic reactions for a rotating rotor $\omega \neq 0$ and $\varepsilon \neq 0$. The following conditions have to be fulfilled:

$$\begin{aligned}
-M\xi\omega^2 &= 0 \\
M\xi\varepsilon &= 0 \\
-J_{xz}\varepsilon + J_{yz}\omega^2 &= 0 \\
-J_{yz}\varepsilon - J_{xz}\omega^2 &= 0
\end{aligned} \tag{15.22}$$

The first two conditions are verified if:

$$\xi = 0, \tag{15.23}$$

meaning that the mass center must be situated on the rotation axis. If this condition applies, the rotor is said to be **statically balanced**.

The last two conditions can be considered a homogeneous system of linear equations in the unknowns ε and ω. Non zero solutions are possible if and only if the determinant $\Delta = J_{xz}^2 + J_{yz}^2$ is equal to zero.

It follows that:

$$J_{xz} = 0; \quad J_{yz} = 0, \tag{15.24}$$

meaning that the rotation axis must be a principal axis of inertia. This is the requirement for rotor to be statically and **dynamically balanced**. In Fig. 15.2 there are three examples of rotors: the (a) rotor shown is unbalanced, the (b) rotor shown is statically balanced and the (c) rotor shown is statically and dynamically balanced.

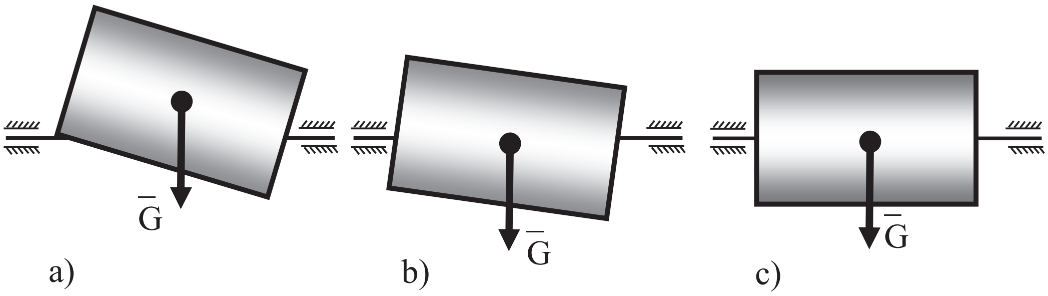



**Fig. 15.2 A rotor can be unbalanced (a), statically balanced (b) or dynamically balanced (c)**

### *15.3. Motion of a rigid body with a fixed point*

A rigid body has a fixed point O which is the origin of a fixed Cartesian frame $O_1x_1y_1z_1$, and another Cartesian frame Oxyz attached to the rigid body (Ox, Oy, Oz are the principal axes of inertia) and $O_1 \equiv O$ (Fig. 15.3).

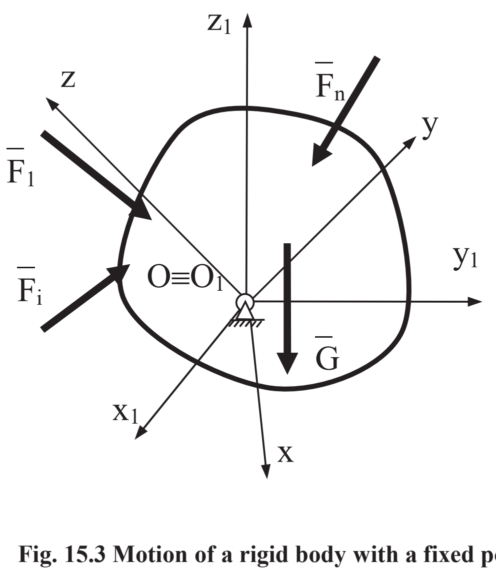



**Fig. 15.3 Motion of a rigid body with a fixed point**

### 15.3.1. The Euler angles

In some applications, the Euler angles are used to simplify the differential equations forms. Three independent rotations can be chosen:

a) A rotation of angle ψ (angle of **precession**) or $\dot{\psi}\overline{k}_1$, of frame (1) around fixed $Oz_1$ axis generates frame (2) (Fig. 15.4 a). The rotation of the new frame can be expressed using matrix [R1] as:

$$\begin{bmatrix} \overline{i}_2 \\ \overline{j}_2 \\ \overline{k}_2 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{i}_1 \\ \overline{j}_1 \\ \overline{k}_1 \end{bmatrix} \overset{not}{=} [R_1] \begin{bmatrix} \overline{i}_1 \\ \overline{j}_1 \\ \overline{k}_1 \end{bmatrix} \tag{15.25}$$

b) A rotation of angle θ (angle of **nutation**) or $\dot{\theta}\overline{i}_2$, of frame (2) around the fixed axis ON (**line of nodes**) generates frame (3) (Fig. 15.4 b). The rotation of the new frame can be expressed using matrix [R2] as:

$$\begin{bmatrix} \overline{i}_3 \\ \overline{j}_3 \\ \overline{k}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \overline{i}_2 \\ \overline{j}_2 \\ \overline{k}_2 \end{bmatrix} \overset{not}{=} [R_2] \begin{bmatrix} \overline{i}_2 \\ \overline{j}_2 \\ \overline{k}_2 \end{bmatrix} \tag{15.26}$$

c) A rotation of angle φ (angle of **local rotation**) or $\dot{\varphi}\overline{k}_3$ around $Oz_3$ axis (Fig. 15.4 c) generates the frame attached to the rigid body. The rotation of the new frame can be expressed using matrix [R3] as:

$$\begin{bmatrix} \overline{i} \\ \overline{j} \\ \overline{k} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{i}_3 \\ \overline{j}_3 \\ \overline{k}_3 \end{bmatrix} \overset{not}{=} [R_3] \begin{bmatrix} \overline{i}_3 \\ \overline{j}_3 \\ \overline{k}_3 \end{bmatrix} \tag{15.27}$$

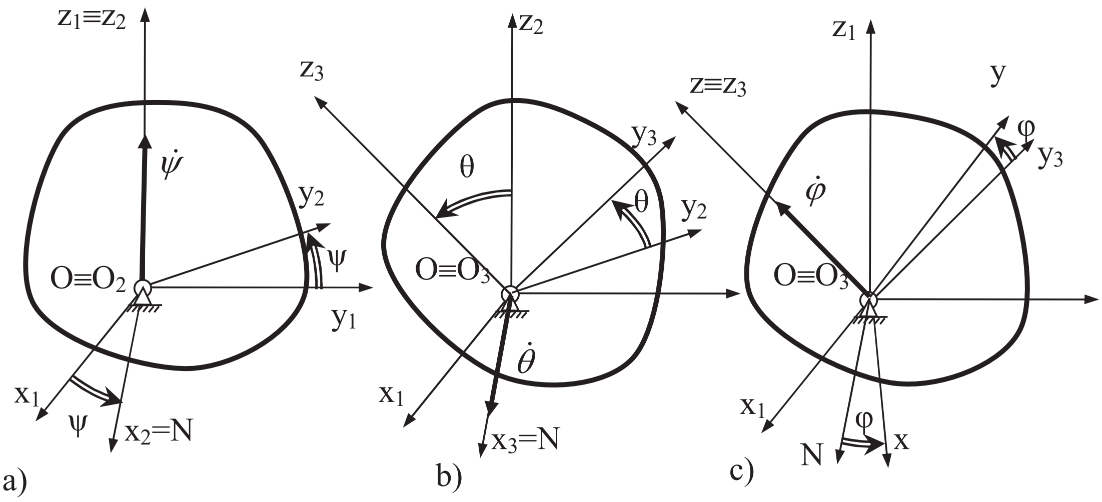



**Fig. 15.4 Rotations defined by the Euler angles**

For example the above angular velocities are projected on the last frame as:

$$\begin{bmatrix}\omega_{\psi x}\\ \omega_{\psi y}\\ \omega_{\psi z}\end{bmatrix}=[R_3][R_2][R_1]\begin{bmatrix}0\\0\\ \dot{\psi}\end{bmatrix}=\begin{bmatrix}\dot{\psi}\sin\theta\sin\varphi\\ \dot{\psi}\sin\theta\cos\varphi\\ \dot{\psi}\cos\theta\end{bmatrix}$$

$$\begin{bmatrix}\omega_{\theta x}\\ \omega_{\theta y}\\ \omega_{\theta z}\end{bmatrix}=[R_3][R_2]\begin{bmatrix}\dot{\theta}\\0\\0\end{bmatrix}=\begin{bmatrix}\dot{\theta}\cos\theta\\ -\dot{\theta}\sin\varphi\\ 0\end{bmatrix} \tag{15.28}$$

$$\begin{bmatrix}\omega_{\varphi x}\\ \omega_{\varphi y}\\ \omega_{\varphi z}\end{bmatrix}=[R_3]\begin{bmatrix}0\\0\\ \dot{\varphi}\end{bmatrix}=\begin{bmatrix}0\\0\\ \dot{\varphi}\end{bmatrix}$$

It follows that the angular velocities $\dot{\psi},\dot{\theta},\dot{\varphi}$ corresponding to these rotations have respectively the directions of $O_1z_1$, $ON$ and $O_3z_3$ (Fig. 15.4). The expressions of the projections of the angular velocity on the axes of movable frame $\omega_x$, $\omega_y$ and $\omega_z$ are the sums of the components obtained above:

$$\begin{aligned}\omega_x&=\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi\\ \omega_y&=\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi\\ \omega_z&=\dot{\psi}\cos\theta+\dot{\varphi}\end{aligned} \tag{15.29}$$

The projections of the Euler rotations on the fixed frame are

$$\begin{bmatrix}\omega_{1\psi x}\\ \omega_{1\psi y}\\ \omega_{1\psi z}\end{bmatrix}=[R_1]^T\begin{bmatrix}0\\0\\ \dot{\psi}\end{bmatrix}=\begin{bmatrix}0\\0\\ \dot{\psi}\end{bmatrix}$$

$$\begin{bmatrix}\omega_{1\theta x}\\ \omega_{1\theta y}\\ \omega_{1\theta z}\end{bmatrix}=[R_1]^T[R_2]^T\begin{bmatrix}\dot{\theta}\\0\\0\end{bmatrix}=\begin{bmatrix}\dot{\theta}\cos\psi\\ \dot{\theta}\sin\psi\\ 0\end{bmatrix} \tag{15.30}$$

$$\begin{bmatrix}\omega_{1\varphi x}\\ \omega_{1\varphi y}\\ \omega_{1\varphi z}\end{bmatrix}=[R_1]^T[R_2]^T[R_3]^T\begin{bmatrix}0\\0\\ \dot{\varphi}\end{bmatrix}=\begin{bmatrix}\dot{\varphi}\sin\theta\sin\psi\\ -\dot{\varphi}\sin\theta\cos\psi\\ \dot{\varphi}\cos\theta\end{bmatrix}$$

Consequently the angular velocity projections on the fixed frame are:

$$\begin{aligned}\omega_{1x}&=\dot{\varphi}\sin\theta\sin\psi+\dot{\theta}\cos\psi\\ \omega_{1y}&=-\dot{\varphi}\sin\theta\cos\psi+\dot{\theta}\sin\psi\\ \omega_{1z}&=\dot{\psi}+\dot{\varphi}\cos\theta\end{aligned} \tag{15.31}$$

### 15.3.2. Dynamics of the rigid body with a fixed point

Denote by $\bar{F}_i\left(i=1,...,n\right)$ the external forces acting on the rigid body, by $\bar{R}$ and $\bar{M}_0$ the resultant force vector and the resultant moment vector of these forces, and by $\bar{\mathsf{R}}$ the reaction in the fixed point O (a smooth spherical joint). The theorems of linear momentum, angular momentum about the fixed point O and respectively the theorem of energy and work are:

$$\frac{d\bar{H}}{dt}=\bar{R}+\bar{\mathsf{R}};\quad \frac{d\bar{K}_0}{dt}=\bar{M}_0;\quad dE=dW. \tag{15.32}$$

The motion of the rigid body is defined by the theorem of angular momentum. The angular momentum, its derivative and the resultant moment vector are:

$$\bar{K}_0=J_1\omega_x\bar{i}+J_2\omega_y\bar{j}+J_3\omega_z\bar{k};$$

$$\frac{d\bar{K}_0}{dt}=\frac{\partial\bar{K}_0}{\partial t}+\bar{\omega}\times\bar{K}$$

$$=J_1\dot{\omega}_x\bar{i}+J_2\dot{\omega}_y\bar{j}+J_3\dot{\omega}_z\bar{k}+\left(\omega_x\bar{i}+\omega_y\bar{j}+\omega_z\bar{k}\right)\times\left(J_1\omega_x\bar{i}+J_2\omega_y\bar{j}+J_3\omega_z\bar{k}\right)= \tag{15.33}$$

$$=\left(J_1\dot{\omega}_x+\left(J_3-J_2\right)\omega_y\omega_z\right)\bar{i}+\left(J_2\dot{\omega}_y+\left(J_1-J_3\right)\omega_z\omega_x\right)\bar{j}+\left(J_3\dot{\omega}_z+\left(J_2-J_1\right)\omega_x\omega_y\right)\bar{k};$$

$$\bar{M}_0=M_{ox}\bar{i}+M_{oy}\bar{j}+M_{oz}\bar{k};$$

The theorem of angular momentum gives us the following equations:

$$\begin{aligned}J_1\dot{\omega}_x+\left(J_3-J_2\right)\omega_y\omega_z&=M_{ox}\\ J_2\dot{\omega}_y+\left(J_1-J_3\right)\omega_z\omega_x&=M_{oy}\\ J_3\dot{\omega}_z+\left(J_2-J_1\right)\omega_x\omega_y&=M_{oz}\end{aligned} \tag{15.34}$$

These equations are called **Euler's equations**. The equations (15.29) and (15.34) represent a system of six first order differential equations with six unknown functions $\omega_x\left(t\right)$, $\omega_y\left(t\right)$, $\omega_z\left(t\right)$, $\varphi\left(t\right)$, $\psi\left(t\right)$, $\theta\left(t\right)$. Note that $M_{ox}$, $M_{oy}$, $M_{oz}$ can depend of $\psi$, $\varphi$, $\theta$ and $t$. There are three cases in which these differential equations can be analytically integrated for any initial conditions.

### 15.3.3. The Euler – Poinsot case

If $M_{ox}=0$, $M_{oy}=0$, $M_{oz}=0$ (a rigid body with a fixed point having no applied forces, or a heavy rigid body with a fixed point at its center of gravity), the equations (15.34) become:

$$\begin{aligned}J_1\dot{\omega}_x+\left(J_3-J_2\right)\omega_y\omega_z&=0\\ J_2\dot{\omega}_y+\left(J_1-J_3\right)\omega_z\omega_x&=0.\\ J_3\dot{\omega}_z+\left(J_2-J_1\right)\omega_x\omega_y&=0\end{aligned} \tag{15.35}$$

Multiplying these equations by $\omega_x(t), \omega_y(t), \omega_z(t)$ respectively and adding the obtained equations, it follows:

$$J_1\omega_x\dot{\omega}_x + J_2\omega_y\dot{\omega}_y + J_3\omega_z\dot{\omega}_z = 0. \tag{15.36}$$

Integrating this last equation, it can be obtained:

$$J_1\omega_x^2 + J_2\omega_y^2 + J_3\omega_z^2 = 2T \tag{15.37}$$

where $T$ is a positive constant representing the energy of the rigid body.

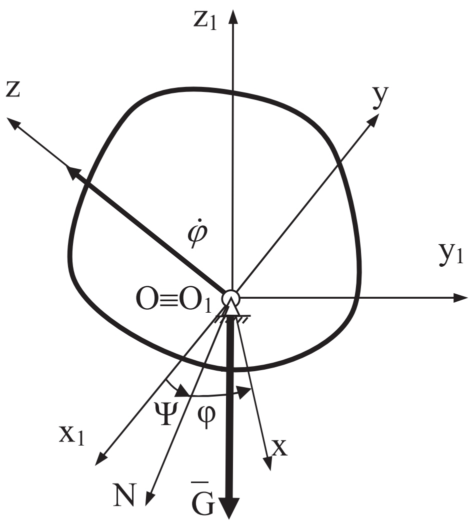



**Fig. 15.5 Euler-Poinsot model of rigid body with a fixed point**

Multiplying now the equations (15.35) respectively by $J_1\omega_x$, $J_2\omega_y$, $J_3\omega$ and adding the obtained equations, a second equation is obtained:

$$J_1^2\omega_x\dot{\omega}_x + J_2^2\omega_y\dot{\omega}_y + J_3^2\omega_z\dot{\omega}_z = 0 \tag{15.38}$$

Integrating this last equation, it follows:

$$J_1^2\omega_x^2 + J_2^2\omega_y^2 + J_3^2\omega_z^2 = K^2 \tag{15.39}$$

where $K^2$ is a positive constant (square of the modulus of the angular momentum).

The equations (15.37)and (15.39) permit to express $\omega_x^2$ and $\omega_z^2$ as functions of $\omega_y^2$:

$$\begin{aligned} \omega_x^2 &= \frac{K^2 - 2TJ_3}{J_1(J_1 - J_3)} - \frac{J_2(J_2 - J_3)}{J_1(J_1 - J_3)}\omega_y^2 \\ \omega_z^2 &= \frac{2TJ_1 - K^2}{J_3(J_1 - J_3)} - \frac{J_2(J_1 - J_2)}{J_3(J_1 - J_3)}\omega_y^2 \end{aligned} \tag{15.40}$$

If the moments of inertia verify the inequality $J_1 < J_2 < J_3$ and this condition can always be assumed, it is easy to verify that the four ratios in (15.40) are all positive:

$$\begin{aligned}
&K^2 - 2TJ_3 = J_1^2\omega_x^2 + J_2^2\omega_y^2 + J_3^2\omega_z^2 - \left(J_1\omega_x^2 + J_2\omega_y^2 + J_3\omega_z^2\right)J_3 \\
&= J_1\left(J_1 - J_3\right)\omega_x^2 + J_2\left(J_2 - J_3\right)\omega_y^2 > 0 \\
&2TJ_1 - K^2 = \left(J_1\omega_x^2 + J_2\omega_y^2 + J_3\omega_z^2\right)J_1 - J_1^2\omega_x^2 - J_2^2\omega_y^2 - J_3^2\omega_z^2 \\
&= J_2\left(J_1 - J_2\right)\omega_y^2 + J_3\left(J_1 - J_3\right)\omega_z^2 > 0
\end{aligned} \tag{15.41}$$

Consequently the square roots will provide real expressions obtained by injecting these angular velocities in the second equation from (15.35):

$$\dot{\omega}_y = \pm\frac{J_3 - J_1}{J_2}\sqrt{\left(\frac{K^2 - 2TJ_3}{J_1\left(J_1 - J_3\right)} - \frac{J_2\left(J_2 - J_3\right)}{J_1\left(J_1 - J_3\right)}\omega_y^2\right)\left(\frac{2TJ_1 - K^2}{J_3\left(J_1 - J_3\right)} - \frac{J_2\left(J_1 - J_2\right)}{J_3\left(J_1 - J_3\right)}\omega_y^2\right)} \tag{15.42}$$

By forcing common factors in the two parentheses, the following change of variable proves to be useful:

$$x = \sqrt{\frac{J_2\left(J_1 - J_2\right)}{2TJ_1 - K^2}}\,\omega_y \Rightarrow d\omega_y = \sqrt{\frac{2TJ_1 - K^2}{J_2\left(J_1 - J_2\right)}}\,dx\,. \tag{15.43}$$

The differential equation (15.42), assuming that the angular velocity increases, such that a positive sign is selected, can be written as

$$\frac{dx}{dt} = \alpha\sqrt{\left(1 - x^2\right)\left(1 - k^2x^2\right)}\,, \tag{15.44}$$

by using the notations:

$$k^2 = \frac{J_2 - J_3}{J_1 - J_2}\frac{2TJ_1 - K^2}{K^2 - 2TJ_3}; \quad \alpha = \sqrt{\frac{J_1 - J_2}{J_1J_2J_3}\left(K^2 - 2TJ_3\right)}\,. \tag{15.45}$$

The integral of (15.44) is:

$$\int_{x_0}^{x}\frac{dx}{\sqrt{\left(1 - x^2\right)\left(1 - k^2x^2\right)}} = \alpha\left(t - t_0\right). \tag{15.46}$$

The left side represents the incomplete elliptic integral of the first kind in the Jacobi's form. Using the substitution $x = \sin\theta$, the left side if this integral becomes the incomplete elliptic integral of the first kind in the Legendre's form:

$$\int_{\Phi_0}^{\Phi}\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} = \alpha\left(t - t_0\right), \tag{15.47}$$

with $x_0 = \sin\Phi_0$, $x = \sin\Phi$. It is known as definition of the elliptic function sine-amplitude *sn(u)*, the following expression:

$$sn(u) = \sin\Phi \quad for \quad \int_0^{\Phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = u\,. \tag{15.48}$$

For given time $t$ and $t_0$, it can be obtained from tables or using a computer, the values of the sinus amplitude for arguments $\alpha t$ and $\alpha t_0$, then $\omega_y$ can be obtanined from (15.43). The other two angular velocities can be deduced from (15.40) and instantaneous rotations can then be thus by deduced for any time t. Alternatively, the differential equation (15.42) can be numerically integrated from the moment $t_0$ at which the constants $T$ and $K$ are determined, to any other moment t.

Poinsot[1] found in 1834 a geometrical interpretation of this motion: The inertia ellipsoid about O rolls on a fixed plane with angular velocity proportional to the OP segment, in which P is the tangency point between the ellipsoid and the fixed plane.

Proof

Considering the equation of the ellipsoid of inertia at the fixed point O:

$$J_1x^2 + J_2y^2 + J_3z^2 = C^2\,, \tag{15.49}$$

the angular velocity $\bar{\omega}$ intersects the ellipsoid in a point P of coordinates:

$$x_O = \lambda\omega_x;\;\; y_O = \lambda\omega_y;\;\; z_O = \lambda\omega_z \tag{15.50}$$

with λ a parameter which can be determined by injecting these coordinates in the ellipsoid equation (15.49). It follows that:

$$\lambda = \frac{C}{\sqrt{J_1\omega_x^2 + J_2\omega_y^2 + J_3\omega_z^2}} = \frac{C}{\sqrt{2T}} = const. \tag{15.51}$$

The coordinates of P are then

$$x_O = \frac{C}{\sqrt{2T}}\omega_x;\;\; y_O = \frac{C}{\sqrt{2T}}\omega_y;\;\; z_O = \frac{C}{\sqrt{2T}}\omega_z\,, \tag{15.52}$$

proving the proportionality between OP vector and angular velocity.
The equation of the tangent plane at P on the ellipsoid is:

$$J_1x_Ox + J_2y_Oy + J_3z_Oz = C^2\,, \tag{15.53}$$

or

$$J_1\omega_x x + J_2\omega_y y + J_3\omega_z z = C\sqrt{2T}\,. \tag{15.54}$$

The normal to this plane has components proportional to $J_1\omega_x;\; J_2\omega_y;\; J_3\omega_z$ which represent the components of the angular momentum. Since the resultant moment

---

[1] *Louis Poinsot (1777–1859) was a French mathematician and physicist. Poinsot was the inventor of geometrical mechanics, showing how a system of forces acting on a rigid body could be resolved into a single force and a couple*

vector is null by hypothesis, it results that the angular momentum is a constant vector. Consequently the plane has a fixed orientation. From analytic geometry, the distance from a fixed point $P(x_o,y_o,z_o)$ and a plane $Ax+By+Cz+D=0$ is

$$d=\frac{Ax_0+By_0+Cz_0+D}{\sqrt{A^2+B^2+C^2}} \tag{15.55}$$

The distance between $O(0,0,0)$ and the tangent plane is also a constant:

$$d=\frac{C^2}{\sqrt{J_1^2x_0^2+J_2^2y_0^2+J_3^2z_0^2}}=\frac{C\sqrt{2T}}{\sqrt{J_1^2\omega_x^2+J_2^2\omega_y^2+J_3^2\omega_z^2}}=\frac{C\sqrt{2T}}{K}=const. \tag{15.56}$$

Therefore the motion of a rigid body with a fixed point O, if $\bar{M}_O=\bar{0}$, can be represented by a rolling motion on a fixed plane of its ellipsoid of inertia determined about the fixed point O.

### 15.3.4. The Lagrange - Poisson case

If $J_1=J_2$ and the unique force acting on the rigid body is its weight $\bar{G}=m\bar{g}$ applied at the mass center $C\ (O,\ O,\ h)$ then $\bar{M}_O$ has the direction of the line of nodes ON, $\left|\bar{M}_O\right|=Mgh\sin\theta$, and the projections of the applied moment on the axes of the moving frame $Oxyz$ are (Fig. 15.6):

$$\begin{aligned}M_{Ox}&=Mgh\sin\theta\cos\varphi\\ M_{Oy}&=-Mgh\sin\theta\sin\varphi\,.\\ M_{Oz}&=0\end{aligned} \tag{15.57}$$

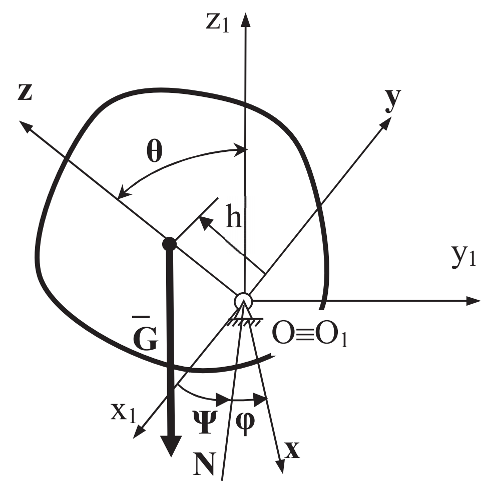



**Fig. 15.6 Lagrange-Poisson model of rigid body with a fixed point**

The equations (15.29) and (15.34) become

$$\begin{cases} \omega_x = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \omega_y = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\ \omega_z = \dot{\psi}\cos\theta + \dot{\varphi} \\ J_1\dot{\omega}_x + \left(J_3 - J_1\right)\omega_y\omega_z = Mgh\sin\theta\cos\varphi \\ J_1\dot{\omega}_y + \left(J_1 - J_3\right)\omega_z\omega_x = -Mgh\sin\theta\sin\varphi \\ J_3\dot{\omega}_z = 0 \end{cases} \tag{15.58}$$

From the last equation it follows that

$$\omega_z = \omega_0 = const. \tag{15.59}$$

Multiplying the last three equations (15.58) by $\omega_x$, $\omega_y$, $\omega_z$ respectively, and afterwards, adding the obtained equations, it follows:

$$J_1\dot{\omega}_x\omega_x + J_1\dot{\omega}_y\omega_y + J_3\dot{\omega}_z\omega_z = \dot{\theta}Mgh\sin\theta. \tag{15.60}$$

The integral of this equation is:

$$\frac{1}{2}J_1\left(\omega_x^2 + \omega_y^2\right) + \frac{1}{2}J_3\omega_z^2 = -Mgh\cos\theta + C. \tag{15.61}$$

or after injecting the angular velocity components on the Ox and Oy axes from (15.58):

$$J_1\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right) + J_3\omega_z^2 = -2Mgh\cos\theta + 2C. \tag{15.62}$$

The angular momentum about the fixed axis $Oz_1$ is constant because the weight is parallel to this axis. It is first necessary to project the angular momentum on the fixed frame $Ox_1y_1z_1$:

$$\begin{bmatrix} K_{1x} \\ K_{1y} \\ K_{1z} \end{bmatrix} = \left[R_1\right]^T\left[R_2\right]^T\left[R_3\right]^T\begin{bmatrix} J_1\omega_x \\ J_1\omega_y \\ J_3\omega_z \end{bmatrix}. \tag{15.63}$$

It follows that the constant component is

$$K_{1z} = J_1\left(\omega_x\sin\varphi + \omega_y\cos\varphi\right)\sin\theta + J_3\omega_3\cos\theta = K_0 = const. \tag{15.64}$$

Replacing the angular velocities components from (15.58), the constant component becomes

$$J_1\dot{\psi}\sin^2\theta + J_3\omega_0\cos\theta = K_0 = const. \tag{15.65}$$

Equations (15.62) and (15.65) can be written as a system of equations as

$$\begin{aligned} &\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta = \alpha - \beta\cos\theta \\ &\dot{\psi}\sin^2\theta = \gamma - \delta\cos\theta \end{aligned} \tag{15.66}$$

with $\alpha, \beta, \gamma, \delta$ four constants easy to identify. Eliminating $\dot{\psi}$ from these equations it follows:

$$\dot{\theta}^2 \sin^2\theta = (\alpha - \beta\cos\theta)\sin^2\theta - (\gamma - \delta\cos\theta)^2. \tag{15.67}$$

By the substitution $\cos\theta = u$, this differential equation becomes:

$$\left(\frac{du}{dt}\right)^2 = (\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2 \overset{not}{=} P(u). \tag{15.68}$$

It follows:

$$dt = \frac{du}{\sqrt{(\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2}}, \tag{15.69}$$

or

$$t = \int_{u_0}^{u} \frac{du}{\sqrt{(\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2}}. \tag{15.70}$$

This is an elliptical integral. It follows that $u = u(t)$ is the inverse of this elliptical integral, which is an elliptical function. The second equation (15.66) can be written with the same substitution:

$$\dot{\psi} = \frac{\gamma - \delta u}{1-u^2}. \tag{15.71}$$

It is now possible to make a qualitative study of the motion. Since $u=\cos\theta$, it follows that $-1 \le u \le +1$ (for $u \to 1$, and $u \to -1$, $\dot{\psi} \to \infty$ if the numerator of (15.71) is not null). The study of the sign of $P(u)$ is summarized in the following two tables. Two cases are possible, because $P(u) \ge 0$ (the motion exists for any time $t$, therefore also for $t = 0$):

Table 1

| u | $-\infty$ | -1 | $u_o$ | 1 | $\infty$ |
|---|---|---|---|---|---|
| P(u) | - | - | + | - | + |

In the first case (Table 1) $P(u)$ has three real roots: $u_1 \in (-1, u_0)$, $u_2 \in (u_0, +1)$ and $u_3 \in (1, \infty)$. The motion of the rigid body is possible only if $u_1 \le u \le u_2$. A sphere with its centre in the fixed point O will be considered in the following plots. The moving axis Oz intersects the sphere in a point which describes paths which are illustrating the investigated cases. If $u_1=\cos\theta_1$ and $u_2=\cos\theta_2$ it follows that $\cos\theta_1 \le \cos\theta_2$, therefore $\theta_1 \ge \theta_2$ and the motion of rigid is possible only if $\theta_2 \le \theta \le \theta_1$ (Fig. 15.7).

Table 2

| u | $-\infty$ | -1 | $u_o$ | 1 | $\infty$ |
|---|---|---|---|---|---|
| P(u) | - | - | 0 | - | + |

In the second case (Table 2), there are three possibilities:

a) $u_1 \in (-1, u_0),\ u_2 = u_0,\ u_3 \in (1, \infty)$

b) $u_1 = u_0,\ u_2 \in (u_0, 1),\ u_3 \in (1, \infty)$

c) $u_1 = u_2 = u_0,\ u_3 \in (1, \infty)$

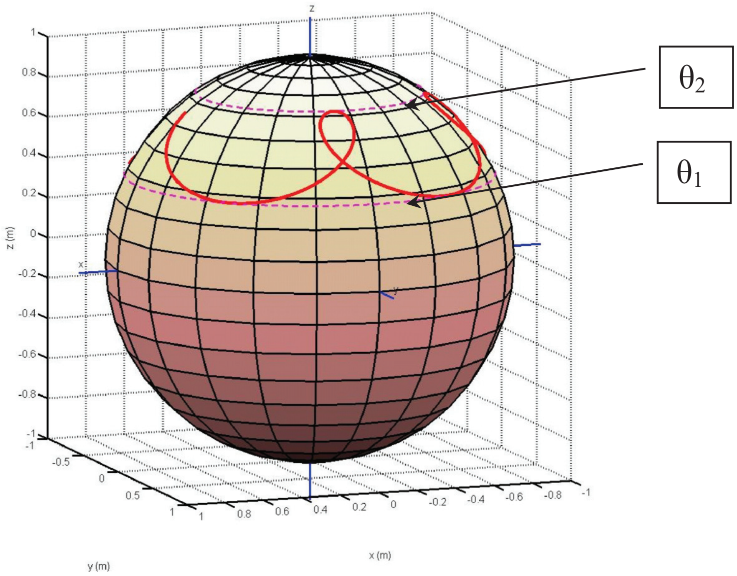



**Fig. 15.7 Motion with loops of the rigid body in the Lagrange- Poisson case**

The path of this point is situated between two horizontal parallel circles $\theta = \theta_1$ and $\theta = \theta_2$. In particular the path is a horizontal circle ( a "parallel") defined by $\theta = \theta_0$. In this last case the motion is called a **regular precession**. Therefore the motion of the rigid body is possible in one of the following cases:

a) only if $\theta_0 \le \theta \le \theta_1$ (e.g. Fig. 15.8);

b) only if $\theta_2 \le \theta \le \theta_0$ (e.g. Fig. 15.9);

c) only if $\theta = \theta_0$ (**regular precession,** Fig. 15.10).

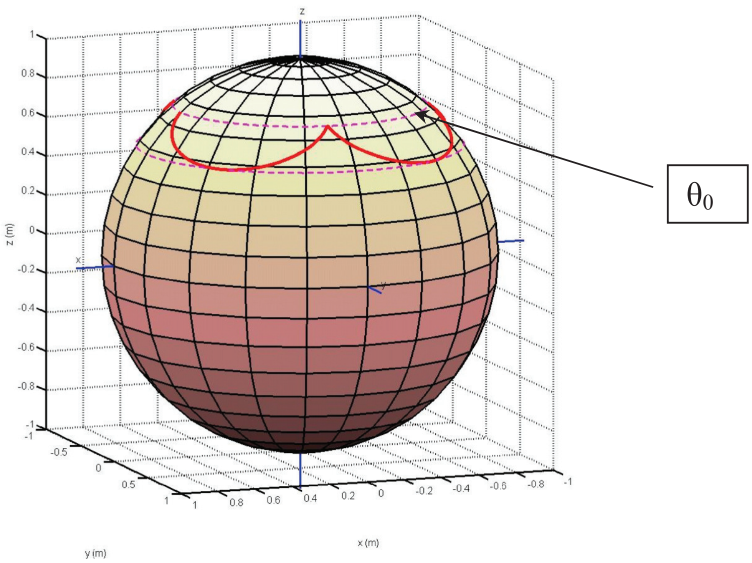



**Fig. 15.8 Cuspidal motion of the rigid body in the Lagrange- Poisson case**

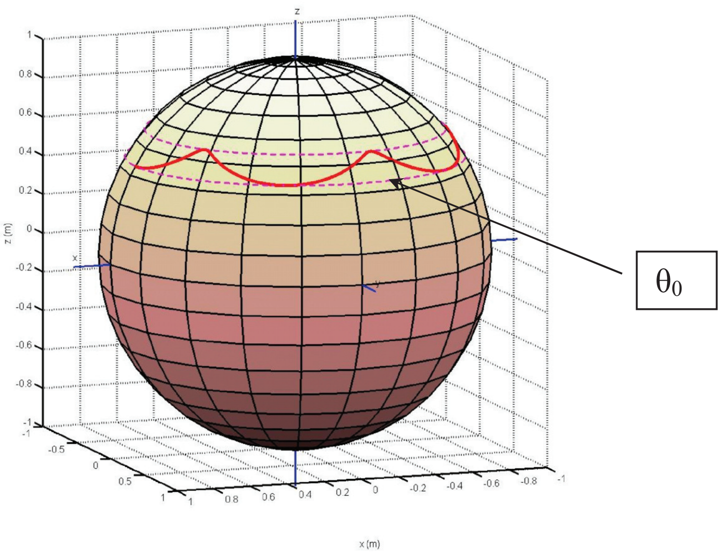



**Fig. 15.9 Wavy motion of the rigid body in the Lagrange- Poisson case**

By examining the relation (15.71) it follows that $\dot{\Psi}=0$ if $\gamma-\delta u=0$, that is $u=\frac{\gamma}{\delta}$. Since $u=\cos\theta$ and $u_1\le u\le u_2$ it follows that $\dot{\Psi}=0$ if and only if

$$u_1 \le \frac{\gamma}{\delta} \le u_2 \tag{15.72}$$

If $\frac{\gamma}{\delta}<u_1$ or $\frac{\gamma}{\delta}>u_2$, then $\dot{\Psi}$ has a constant sign. The motion of point M resemble the motion of a reversed spherical pendulum (Fig. 15.9). If the condition (15.72) is accomplished, then the motion of M has a path with loops (Fig. 15.7).

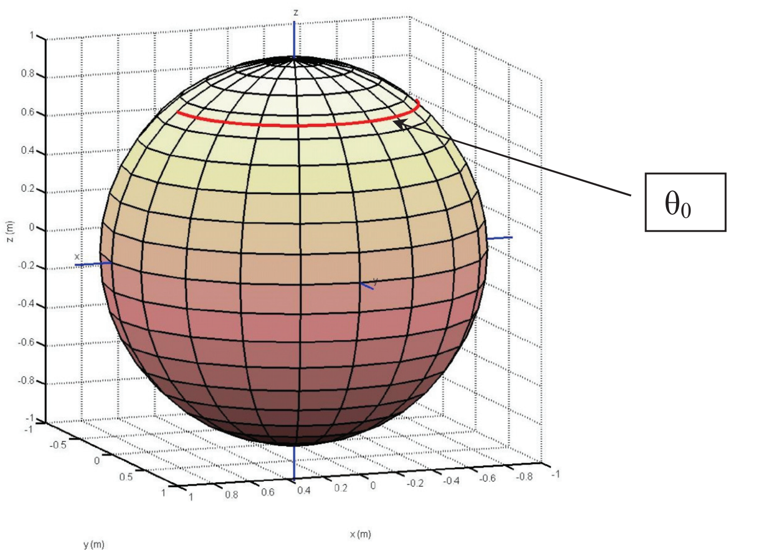



**Fig. 15.10 The regular precession motion of the rigid body in the Lagrange- Poisson case**

If $\frac{\gamma}{\delta}=u_1$, then for $u = u_1$, the simultaneous results $\dot{\theta}=0$ and $\dot{\Psi}=0$ are valid. The motion of M has a cuspidal path (similar to Fig. 15.8).

### 15.3.5. The Sofia Kovalevskaya case

The following hypothesis are assumed: $J_1 = J_2 = 2J_3$, the centre of gravity is situated on the $Ox$ - axis and the only force acting on the rigid body is its weight $M\overline{g}$. Let $\alpha_3, \beta_3, \gamma_3$ be the projections of the unit vector $\overline{k}_1$ of the fixed axis $O_1z_1$ on the axes of the moving Cartesian frame $Oxyz$. The position vector of the mass centre C is $\overline{\rho}=\xi\overline{i}$. It follows that:

$$\overline{M}_O=\overline{\rho}\times M\overline{g}=\xi\overline{i}\times Mg\left(-\alpha_3\overline{i}-\beta_3\overline{j}-\gamma_3\overline{k}\right)=Mg\xi\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 1 & 0 & 0\\ -\alpha_3 & -\beta_3 & -\gamma_3\end{vmatrix} \tag{15.73}$$

$$=Mg\xi\left(\gamma_3\overline{j}-\beta_3\overline{k}\right)$$

Euler's equations (15.34) become, after injecting the components of the force's moments and the moments of inertia according to the hypothesis:

$$\begin{cases} J_1\dot{\omega}_x+\left(\dfrac{J_1}{2}-J_1\right)\omega_y\omega_z=0\\ J_1\dot{\omega}_y+\left(J_1-\dfrac{J_1}{2}\right)\omega_z\omega_x=Mg\xi\gamma_3\\ \dfrac{J_1}{2}\dot{\omega}_z=-Mg\xi\beta_3\end{cases} \tag{15.74}$$

Using the notations

$$c=2\frac{Mg\xi}{J_1};\quad \alpha=c\alpha_3;\quad \beta=c\beta_3;\quad \gamma=c\gamma_3, \tag{15.75}$$

the last equations can be simplified to:

$$\begin{cases} 2\dot{\omega}_x-\omega_y\omega_z=0\\ 2\dot{\omega}_y+\omega_z\omega_x=\gamma\,.\\ \dot{\omega}_z=-\beta\end{cases} \tag{15.76}$$

The second equation multiplied by $i=\sqrt{-1}$ is added to the first one, to yield a complex expression:

$$2\left(\dot{\omega}_x+i\dot{\omega}_y\right)=-i\omega_z\left(\omega_x+i\omega_y\right)+ic\gamma_3 \tag{15.77}$$

Since $O_1z_1$ is a fixed axis, then $\overline{k}_1=const$. It follows that its derivative in a moving frame *Oxyz* is null :

$$\frac{d\overline{k}_1}{dt}=\frac{\partial\overline{k}_1}{\partial t}+\overline{\omega}\times\overline{k}_1=\left(\dot{\alpha}_3\overline{i}+\dot{\beta}_3\overline{j}+\dot{\gamma}_3\overline{k}\right)+\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ \omega_x & \omega_y & \omega_z\\ \alpha_3 & \beta_3 & \gamma_3\end{vmatrix}$$
$$=\left(\dot{\alpha}_3+\omega_y\gamma_3-\omega_z\beta_3\right)\overline{i}+\left(\dot{\beta}_3+\omega_z\alpha_3-\omega_x\gamma_3\right)\overline{j}+\left(\dot{\gamma}_3+\omega_x\beta_3-\omega_y\alpha_3\right)\overline{k}=\overline{0} \qquad (15.78)$$

The direct consequence is the following system of differential equations:

$$\begin{cases}\dot{\alpha}_3+\omega_y\gamma_3-\omega_z\beta_3=0\\ \dot{\beta}_3+\omega_z\alpha_3-\omega_x\gamma_3=0\\ \dot{\gamma}_3+\omega_x\beta_3-\omega_y\alpha_3=0\end{cases} \qquad (15.79)$$

In a similar manner, the first two equations can be cast into complex form:

$$\dot{\alpha}_3+i\dot{\beta}_3=-i\omega_z\left(\alpha_3+i\beta_3\right)+i\gamma_3\left(\omega_x+i\omega_y\right) \qquad (15.80)$$

From this equation and equation (15.77), can be eliminated $\gamma_3$ as follows :

$$2\left(\dot{\omega}_x+i\dot{\omega}_y\right)\left(\omega_x+i\omega_y\right)-c\left(\dot{\alpha}_3+i\dot{\beta}_3\right)=ic\omega_z\left(\alpha_3+i\beta_3\right)-i\omega_z\left(\omega_x+i\omega_y\right)^2$$
$$=-i\omega_z\left[\left(\omega_x+i\omega_y\right)^2-c\left(\alpha_3+i\beta_3\right)\right] \qquad (15.81)$$

or

$$\frac{2\left(\dot{\omega}_x+i\dot{\omega}_y\right)\left(\omega_x+i\omega_y\right)-c\left(\dot{\alpha}_3+i\dot{\beta}_3\right)}{\left(\omega_x+i\omega_y\right)^2-c\left(\alpha_3+i\beta_3\right)}=-i\omega_z \qquad (15.82)$$

It can be remarked that the numerator is the time derivative of the denominator. Consequently, the last expression can be written as:

$$\frac{d}{dt}\left\{\ln\left[\left(\omega_x+i\omega_y\right)^2-c\left(\alpha_3+i\beta_3\right)\right]\right\}=-i\omega_z \qquad (15.83)$$

The passage to the complex form can be done again using $-i$ instead of $i$, so that the following expression will be obtained:

$$\frac{d}{dt}\left\{\ln\left[\left(\omega_x-i\omega_y\right)^2-c\left(\alpha_3-i\beta_3\right)\right]\right\}=i\omega_z. \qquad (15.84)$$

Adding the last two expressions, the result is

$$\ln\left\{\left[\left(\omega_x+i\omega_y\right)^2-c\left(\alpha_3+i\beta_3\right)\right]\left[\left(\omega_x-i\omega_y\right)^2-c\left(\alpha_3-i\beta_3\right)\right]\right\}=const. \qquad (15.85)$$

or

$$\left[\left(\omega_x+i\omega_y\right)^2-c\left(\alpha_3+i\beta_3\right)\right]\left[\left(\omega_x-i\omega_y\right)^2-c\left(\alpha_3-i\beta_3\right)\right]=const. \qquad (15.86)$$

This is the **prime integral** established by Sophia Kovalevskaia (1850-1891). There are also two other prime integrals: the integral of the energy (the total energy of the system is constant) and the prime integral of the angular momentum with respect to the fixed axis $O_1z_1$ ($K_{z1}$ is constant because $M_{1z} = 0$, since the force $Mg$ is parallel to $O_1z_1$). Note that

$$\alpha^2 + \beta^2 + \gamma^2 = c^2 = 4\left(\frac{Mg\xi}{J_1}\right)^2 = const. \qquad (15.87)$$

Therefore, there are four prime integrals for six unknown functions $\omega_x, \omega_y, \omega_z, \alpha, \beta, \gamma$, which can be obtained by integral calculus, for any given initial conditions.

### *15.4. Gyroscope*

The gyroscope is an axially symmetric rigid body ($J_1=J_2=A;\ J_3=C$) suspended by two rings with mutually perpendicular axes, such that the mass center remains in a fixed position relative to a fixed frame. It is a particular case of the Euler-Poinsot case.
Using the Euler angles, the following six differential equations govern the motion of the gyroscope:

$$\begin{aligned}
&\omega_x = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\
&\omega_y = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\
&\omega_z = \dot{\psi}\cos\theta + \dot{\varphi} \\
&A\dot{\omega}_x + (C - A)\omega_y\omega_z = 0 \\
&A\dot{\omega}_y - (C - A)\omega_z\omega_x = 0 \\
&J_3\dot{\omega}_z = 0
\end{aligned} \qquad (15.88)$$

From the last equation it follows $\omega_z = \Omega = const$. Injecting this result into the fourth and fifth equations and denoting $p = \frac{C-A}{A}\Omega$, one gets

$$\begin{aligned}
&\dot{\omega}_x + p\omega_y = 0 \\
&\dot{\omega}_y - p\omega_x = 0
\end{aligned}. \qquad (15.89)$$

By eliminating $\omega_y$ a second order homogeneous differential equation is obtained:

$$\ddot{\omega}_x + p^2\omega_x = 0, \qquad (15.90)$$

for which, using integration constants $\varepsilon, \alpha$, the solution can be written as:

$$\omega_x = \varepsilon\sin(pt - \alpha). \qquad (15.91)$$

The other component of the angular velocity is obtained from (15.89):

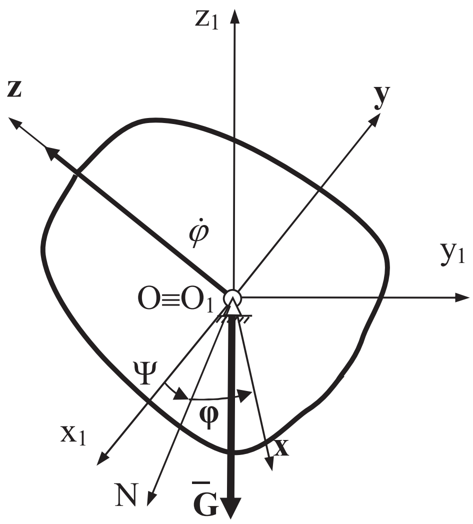



**Fig. 15.11 The perfectly balanced gyroscope**

$$\omega_y = -\varepsilon\cos(pt-\alpha). \tag{15.92}$$

The first three equations from (15.88) can be written using the expressions for the angular velocities as:

$$\begin{aligned}
&\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi = \varepsilon\sin(pt-\alpha) \\
&\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi = -\varepsilon\cos(pt-\alpha) \\
&\dot{\psi}\cos\theta + \dot{\varphi} = \Omega
\end{aligned} \tag{15.93}$$

For some initial conditions, the motion of the rigid body is a **regular precession**, defined by $\theta = \theta_0 = const$. that is constant nutation angle. Supposing that these initial conditions are accomplished, the first two equations become:

$$\begin{cases}\dot{\psi}\sin\theta_0\sin\varphi = \varepsilon\sin(pt-\alpha) \\ \dot{\psi}\sin\theta_0\cos\varphi = -\varepsilon\cos(pt-\alpha)\end{cases}. \tag{15.94}$$

Summing the squared above equations term by term, one gets:

$$\dot{\psi}^2\sin^2\theta_0 = \varepsilon^2 = const. \Rightarrow \dot{\psi}\sin\theta_0 = \pm\varepsilon. \tag{15.95}$$

Consequently another characteristic of the precession motion is $\dot{\psi} = const.$
After derivation with respect to time of both equations from (15.94), a simplification by ε and taking into account the last result, it follows:

$$\begin{cases}\pm\dot{\varphi}\cos\varphi = p\cos(pt-\alpha) \\ \pm\dot{\varphi}\sin\varphi = p\sin(pt-\alpha)\end{cases} \tag{15.96}$$

By coefficient identification it follows that only $\dot{\varphi} = +p$ is the acceptable solution, which means the gyroscope has a constant precession angular velocity and

$$\varphi = pt - \alpha \,. \tag{15.97}$$

Injecting this result in the last equation (15.93), it will be obtained

$$\dot{\psi} \cos\theta_0 = \Omega - p \,. \tag{15.98}$$

Taking the correct sign in (15.95) and dividing term by term that equation by the terms of the last equation, one gets:

$$\tan\theta_0 = \frac{\varepsilon}{\Omega - \dfrac{C-A}{A}\Omega} = \frac{A}{2A-C}\frac{\varepsilon}{\Omega} \,. \tag{15.99}$$

Usually $\Omega \gg \varepsilon$ and so the nutation angle $\theta_0$ is very small, which proves that for small perturbations, the rotation axis of the gyroscope remains in the vicinity of the initial direction of rotation. In other words, the rotations axis is **stable**. On the contrary for small $\Omega$ the nutation angle is large and the motion is unstable. This property of stability was used in gyroscopes which maintain their axis of rotation imposed at take-off, during the arbitrary motion of an airplane. In this way the pilot is informed about the position of the airplane relative to the vertical axis at take-off (gyro-horizon) or about the North direction at take-off (gyro-compass).

### *15.5. Gyroscopic effect*

A perfect gyroscope must have the mass center in the center of rotation, but this condition is accomplished only to a certain degree of accuracy. The mass center is however supposed to be on the rotation axis at a distance *h* from the fixed point (spherical hinge). This situation corresponds to the Lagrange - Poisson case.

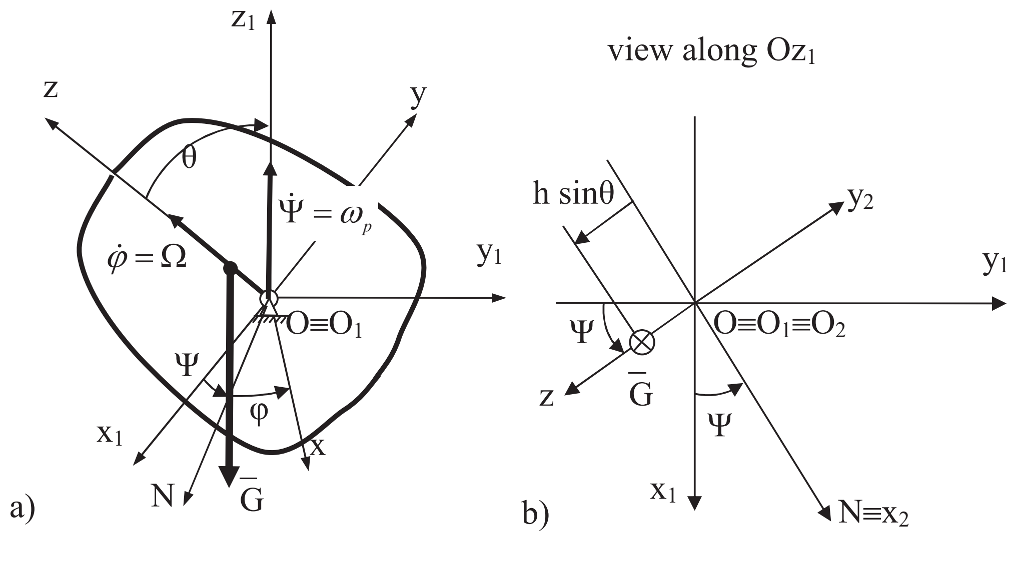



**Fig. 15.12 Gyroscope with *Mgh* unbalance (a), projections of axes on the $O_1x_1y_1$ plane (b)**

In the absence of gyroscope rotation, obviously the weight makes the body moves such that the mass center will oscillate about the lowest possible position ($-h\bar{k}$). However, if the gyroscope rotates with high angular velocity $\dot{\varphi}=\Omega$ about the Oz axis, it will not fall, but for some initial conditions will turn around the $Oz_1$ axis in a motion of regular precession. This motion corresponds to the following parameters: constant nutation angle $\theta=\theta_0=const$. and constant precession angular velocity $\dot{\Psi}=\omega_p=const$. Without loss of generality, it can be assumed that the rotation angle is null at *t=0*, so that from $\dot{\varphi}=\Omega$ it follows $\varphi=\Omega t$.
Replacing these parameters into the system of equations for the Lagrange - Poisson case (15.58), it becomes:

$$\begin{cases} \omega_x = \omega_p \sin\theta_0 \sin\Omega t \\ \omega_y = \omega_p \sin\theta_0 \cos\Omega t \\ \omega_z = \omega_p \cos\theta_0 + \Omega \\ A\dot{\omega}_x + (C-A)\omega_y\omega_z = Mgh\sin\theta\cos\varphi \\ A\dot{\omega}_y - (C-A)\omega_z\omega_x = -Mgh\sin\theta\sin\varphi \\ C\dot{\omega}_z = 0 \end{cases} \tag{15.100}$$

The derivatives with respect to time of the first three equations are:

$$\begin{aligned} \dot{\omega}_x &= \omega_p \Omega \sin\theta_0 \cos\Omega t \\ \dot{\omega}_y &= -\omega_p \Omega \sin\theta_0 \sin\Omega t \\ \dot{\omega}_z &= 0 \end{aligned} \tag{15.101}$$

and these expression will be injected in the last three equations from (15.100), in order to obtain after simplification by the harmonic functions in $\Omega t$, the following:

$$\begin{aligned} & Mgh\sin\theta_0 - C\Omega\omega_p \sin\theta_0\left[1+\frac{C-A}{C}\frac{\omega_p}{\Omega}\cos\theta_0\right]=0 \\ & Mgh\sin\theta_0 - C\Omega\omega_p \sin\theta_0\left[1+\frac{C-A}{C}\frac{\omega_p}{\Omega}\cos\theta_0\right]=0 \\ & 0=0 \end{aligned} \tag{15.102}$$

The second terms in each of the first two equalities can be interpreted as components of a moment produced by the moving gyroscope called **gyroscopic moment** which "equilibrates" the moment of the weight. Following the definitions of the rotations angles, the weight vector $\bar{G}$ crosses the $O_1x_1y_1$ plane in a point belonging to the $O_1N$ axis. The moment of the weight projected on the fixed frame is:

$$\bar{M}_O = \begin{vmatrix} \bar{i}_1 & \bar{j}_1 & \bar{k}_1 \\ h\sin\theta_0\sin\Psi & -h\sin\theta_0\cos\Psi & h\cos\theta_0 \\ 0 & 0 & -Mg \end{vmatrix} \tag{15.103}$$
$$= Mgh\sin\theta_0\cos\Psi\bar{i}_1 + Mgh\sin\theta_0\sin\Psi\bar{j}_1$$

According to the rotation matrix [R$_1$] from (15.25), this vector can be projected on the moving frame $O_2x_2y_2$ as

$$\begin{bmatrix} M_{Ox_2} \\ M_{Oy_2} \\ M_{Ox_2} \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Mgh\sin\theta_0\cos\Psi \\ Mgh\sin\theta_0\sin\Psi \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} Mgh\sin\theta_0 \\ 0 \\ 0 \end{bmatrix} = Mgh\sin\theta_0\bar{i}_2 \tag{15.104}$$

It can be remarked that $\bar{i}_2 = \bar{k}_1 \times \bar{k}$ (Fig. 15.12b), so that

$$\Omega\omega_p\sin\theta_0\bar{i}_2 = \bar{\omega}_p \times \bar{\Omega}. \tag{15.105}$$

Since the moment of the weight is equal in magnitude but opposed in orientation to the gyroscopic moment, it follows that the gyroscopic moment vector must be:

$$\bar{M}_g = -C\left[1 + \frac{C-A}{C}\frac{\omega_p}{\Omega}\cos\theta_0\right]\Omega\omega_p\sin\theta_0\bar{i}_2 \tag{15.106}$$

Using (15.105) in the last formula, it can be written that the gyroscopic moment as a vector is perpendicular on the plane defined by the precession velocity vector $\bar{\omega}_p$ and the rotation velocity vector $\bar{\Omega}$:

$$\bar{M}_g = C\left[1 + \frac{C-A}{C}\frac{\omega_p}{\Omega}\cos\theta_0\right]\left(\bar{\Omega} \times \bar{\omega}_p\right). \tag{15.107}$$

Easier to apply in applications is to neglect the second term in the sum above since $\omega_p \ll \Omega$, making shorter the above formula:

$$\bar{M}_g \cong C\bar{\Omega} \times \bar{\omega}_p. \tag{15.108}$$

It must be emphasized that the gyroscopic moment can be considered as an active moment applied on the body which tries to change the orientation of the gyroscope rotation axis.

Examples

1) A train is following a curve of the railway (Fig. 15.13). If *R* is the radius of the curve, *r* the radius of its wheels and *v* the velocity of the vehicle, it can be deduced:

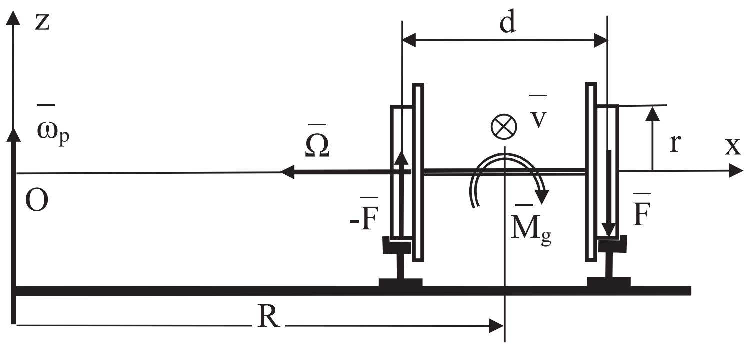



**Fig. 15.13 A train in a curve**

The angular velocity of rotation is $\Omega = \frac{v}{r}$ for an equivalent rigid body made of the two wheels, with O as fixed point. The rotation axis changes its direction, with an angular velocity $\omega_p = \frac{v}{R}$. Supposing that C is the mechanical moment of inertia of the two wheels and the shaft connecting them, determined about the Oz axis, the gyroscopic moment acting on the rails is $\overline{M}_g \cong C\overline{\Omega} \times \overline{\omega}_p$. This couple of forces is adding in fact two forces F to the rails separated by a distance *d*:

$$|F| = \frac{|M_g|}{d} = \frac{Cv^2}{rRd}$$

The gyroscopic effect (Fig. 15.13) is increasing the pressure on the external rail and the pressure on the internal one decreases.

2) A small airplane is engaging in a turn to the left of radius R, with a constant velocity v. The single engine rotates with N rotations per minute. It has a mechanical moment of inertia of the propeller and the rotating parts of the engine (especially for turbo-engines) of C, determined about the plane longitudinal axis passing through the mass center. Determine the effect of the gyroscopic moment on the flight path.

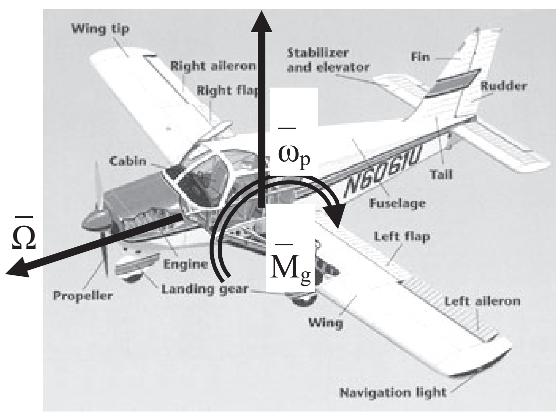



**Fig. 15.14 A small airplane turning left (www.nasa.gov)**

The angular velocity of the engine is $\Omega = \frac{N\pi}{30}\left(rad/s\right)$ and as vector is usually oriented as indicated in Fig. 15.14 The angular velocity of the plane is $\omega_p = \frac{v}{R}$.

The gyroscopic moment is $\overline{M}_g \cong C\overline{\Omega} \times \overline{\omega}_p$ acting as depicted in Fig. 15.14, making the airplane to climb. On the contrary, turning right makes the airplane to dive, so that the pilot must take the corresponding counter-actions to keep the plane level.

### *15.6. General motion of the rigid body*

An arbitrary rigid body is supposed to have a general motion under the action of forces $\overline{F}_i$ (external applied forces and possibly reactions). Let $O_1x_1y_1z_1$ be a fixed Cartesian frame, O the mass center of the rigid body an *Oxyz* a movable Cartesian frame, attached to the body. The frame axes Ox, Oy, Oz are the principal axes of inertia in the point O. The motion of the rigid body can be decomposed in two motions:

a) the motion of its center mass ;
b) the relative motion of the rigid body about its center of mass .

The first motion can be determined using the center of mass theorem. It follows that:

$$\begin{aligned} M\ddot{\xi}_1 &= X_1 \\ M\ddot{\eta}_1 &= Y_1 \\ M\ddot{\zeta}_1 &= Z_1 \end{aligned} \tag{15.109}$$

where $\xi_1$ , $\eta_1$ , $\zeta_1$ are the coordinates of the mass center with respect to the fixed Cartesian frame from $\overline{R} = \sum_{i=1}^{n} \overline{F}_i = X_1\overline{i} + Y_i\overline{j} + Z_i\overline{k}$ are the projections of the resultant vector of all applied and reaction forces on the axis of this frame.

The second motion is a motion of a rigid body about a fixed point O, which can be determined using Euler's equations:

$$\begin{aligned} \omega_x &= \psi \sin\theta \sin\varphi + \theta \cos\varphi \\ \omega_y &= \psi \sin\theta \cos\varphi - \theta \sin\varphi \\ \omega_z &= \psi \cos\theta + \varphi \\ J_1\dot{\omega}_x + (J_3 - J_2)\omega_y\omega_z &= M_{0x} \\ J_2\dot{\omega}_y + (J_1 - J_3)\omega_z\omega_x &= M_{0y} \\ J_3\dot{\omega}_z + (J_2 - J_1)\omega_x\omega_y &= M_{0z} \end{aligned} \tag{15.110}$$

where $\psi$, $\varphi$, $\theta$ are Euler's angles and $M_{0x}$, $M_{0y}$, $M_{0z}$ represent the projections of the resultant moment $\overline{M}_0$ on the axes of the Cartesian frame Oxyz .
In general $X_1$, $Y_1$, $Z_1$, $M_{0x}$, $M_{0y}$, $M_{0z}$ are functions of $\xi_1$, $\eta_1$, $\zeta_1$, $\psi, \varphi, \theta$. Therefore (15.109) and (15.110) represent a system that can not be separated in independent equations. Only in particular cases the decomposition in independent differential equations is possible.

Example
Determine the motion of a rolling homogeneous circular disk of radius *R* and mass *M* on an inclined rough plane of angle *α* with horizontal direction. The sliding friction coefficient is *μ*. Rolling friction is at first ignored, then included with rolling friction coefficient *s*. Discuss motion as function of parameter *α*.

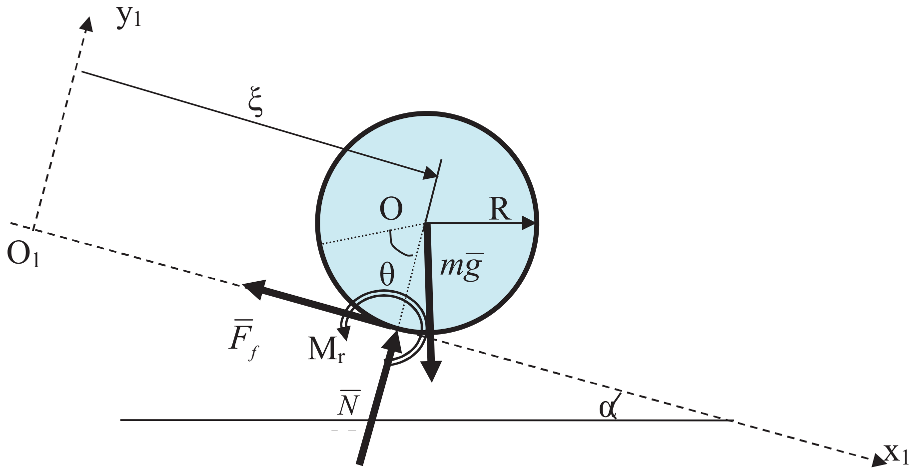



**Fig. 15.15 Rolling disk**

1) The free body diagram is shown in Fig. 15.15. Besides the weight are plotted the normal reaction and the friction force $F_f$ denoted also T in some books. The rolling friction moment $M_r$ is ignored at this stage. The linear momentum theorem for the center of mass can be written:

$$\begin{aligned} &(Ox_1): \; M\ddot{\xi} = Mg\sin\alpha - F_f \\ &(Oy_1): \; M\ddot{\eta} = N - Mg\cos\alpha \end{aligned}. \tag{15.111}$$

The relative motion about the center of mass O is a rotation of angle θ. The theorem of angular momentum about the Oz axis (perpendicular on the figure) can be written

$$J_0\ddot{\theta} = F_f R . \tag{15.112}$$

The following geometrical conditions are obvious, for pure rolling motion:

$$\eta = R; \quad \xi = R\theta . \tag{15.113}$$

Solving the system of differential equations (15.111) + (15.112) it follows that:

$$\ddot{\theta} = \frac{MgR\sin\alpha}{J + MR^2} \quad ; \quad \ddot{\xi} = \frac{MgR^2\sin\alpha}{J + MR^2}$$
$$N = Mg\cos\alpha \quad ; \quad F_f = \frac{MgJ}{J_0 + MR^2}\sin\alpha \quad . \qquad (15.114)$$

Since the disk is homogeneous $J_0 = \frac{MR^2}{2}$ and the above expressions become

$$\ddot{\theta} = \frac{2g}{3R}\sin\alpha; \; \ddot{\xi} = \frac{2}{3}g\sin\alpha; \;\; N = Mg\cos\alpha; \;\; F_f = \frac{1}{3}Mg\sin\alpha . \qquad (15.115)$$

Consequently the mass center has a uniformly accelerated linear motion and the disk rotates around its center uniformly accelerated.
The assumed rolling motion is possible only if $F_f < \mu N$. The condition resulting from this condition is :

$$\frac{1}{3}Mg\sin\alpha < \mu Mg\cos\alpha \quad \text{or} \quad \tan\alpha < 3\mu . \qquad (15.116)$$

If $\tan\alpha > 3\mu$, the motion of the circular disk is a combination of rolling and sliding depending on the initial conditions.

2) If the rolling friction moment is to be accounted for, it will be added as $M_r = sN$ (Fig. 15.15). The equation (15.112) becomes:

$$J\ddot{\theta} = F_f R - M_r , \qquad (15.117)$$

and the expressions (15.115) become:

$$\ddot{\theta} = \frac{2g}{3R}(\sin\alpha - \frac{s}{R}\cos\alpha); \;\; \ddot{\xi} = \frac{2g}{3}(\sin\alpha - \frac{s}{R}\cos\alpha);$$
$$N = Mg\cos\alpha; \qquad F_f = Mg\left(\frac{\sin\alpha}{3} + \frac{2s}{3R}\cos\alpha\right) \qquad (15.118)$$

The necessary condition to avoid sliding: $|F_f| < \mu N$ becomes:

$$\tan\alpha < 3\mu - \frac{2s}{R} . \qquad (15.119)$$

3) The disk is assumed to be at rest for t=0, which implies $\ddot{\theta} = 0$; $\ddot{\xi} = 0$. The differential equations (15.111) and (15.117) become equilibrium conditions:

$$Mg\sin\alpha - F_f = 0; \;\; N - Mg\cos\alpha = 0; \;\; F_f R - M_r = 0; \;\; M_r = sN \qquad (15.120)$$

It follows that for equilibrium:

$$N = Mg\cos\alpha; \;\; F_f = Mg\sin\alpha; \;\; M_r = MgR\sin\alpha;$$
$$MgR\sin\alpha = sMg\cos\alpha \qquad (15.121)$$

From the last equation representing limit equilibrium with rolling tendency:

$$\tan\alpha = \frac{s}{R}. \tag{15.122}$$

The condition of no sliding $F_f < \mu N$ becomes:

$$Mg\sin\alpha < \mu Mg\cos\alpha \quad \Rightarrow \quad \tan\alpha < \mu \tag{15.123}$$

From the last two relations it follows the limit condition of rolling tendency without sliding:

$$\frac{s}{R} < \mu . \tag{15.124}$$

The consequence is a condition for the $\alpha$ angle, such that the disk begins to roll without sliding, from its initial position:

$$\frac{s}{R} < \tan\alpha < \mu . \tag{15.125}$$

4) Therefore, if $\frac{s}{R} < \mu$ the pure rolling (rolling without sliding) of the circular disk is possible if:

$$\frac{s}{R} < \tan\alpha < 3\mu - \frac{2s}{R} \tag{15.126}$$

For angles α below the lower limit, the disk will remain at rest if it was initially at rest. For angles above the upper limit, the disk will roll and slide in a combination of these two motions, depending on the initial conditions.

# 16. PERCUSSIVE MOTIONS

## 16.1. Preliminaries. Basic concepts

The governing equation for a particle moving under the action of a force $\overline{F}$ is:

$$m\overline{a} = \overline{F}\left(\overline{r}, \overline{v}, t\right). \tag{16.1}$$

This equation can be used if $\overline{F}$ is defined as a function of position, velocity and time if not a constant. There are motions with almost instantaneous change in velocity, but no change in particle position. Such phenomenon arises when a very large force acts on the particle during a very short lapse of time. Considering the action of the force between $t_1$ and $t_2$, having a time lapse $t_2 - t_1$ very small, it can be defined the **percussion** as:

$$\overline{P} = \int_{t_1}^{t_2} \overline{F} dt . \tag{16.2}$$

The force $\overline{F}$ is called to be an **impulsive** one. An example of impulsive motion is a collision between two spherical bodies which produces large forces acting for a short duration. During a collision there can be identified two periods: a period of compression and a period of restitution (expansion).

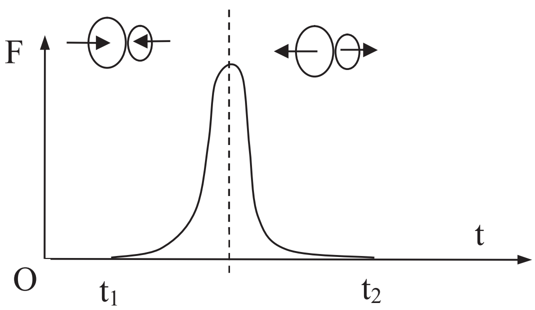



**Fig. 16.1 Percusion and the two periods: compression and restitution**

During the period of compression in the given example, the centers of two spherical balls are approaching each other and then the balls start to regain their spherical shapes, pressing against each other until they separate.

The normal percussion $P_{nr}$ during restitution is smaller than the normal percussion $P_{nc}$ during compression, defined by a ratio k, called **coefficient of restitution**:

$$P_{nr} = k\, P_{nc} \tag{16.3}$$

These two percussions $P_{nr}$ and $P_{nc}$ act on a given body in the same direction and sense and are normal to the tangent plane at the point of collision common to the two bodies. The coefficient of restitution verifies the inequalities:

$$0 \le k \le 1 . \tag{16.4}$$

If $k = 1$ the collision is said to be **perfectly elastic** and if $k = 0$ it is **perfectly plastic**, but none of these cases corresponds exactly to real facts.
The impulsive forces are very large and compared to them, the usual external forces can be neglected. The displacements of two bodies during the very short interval of percussion can also be neglected. These are the hypothesis used in the following

## *16.2. General theorems of dynamics in the case of impulsive motions*

### 16.2.1. The theorem of linear momentum

From the general form of the theorem:

$$\frac{d\overline{H}}{dt} = \sum_{i=1}^{n} \overline{F}_i \,, \tag{16.5}$$

in which $\overline{F}_i$, *i=1,…,n* are only the impulsive external forces, it can be written successively :

$$d\overline{H} = \sum_{i=1}^{n} \overline{F}_i dt \;\Rightarrow \int_{t_1}^{t_2} d\overline{H} = \int_{t_1}^{t_2} \sum_{i=1}^{n} \overline{F}_i dt = \sum_{i=1}^{n} \left( \int_{t_1}^{t_2} \overline{F}_i dt \right) = \sum_{i=1}^{n} \overline{P}_i \tag{16.6}$$

In fact the linear momentum undertake a sudden change, so that the differential corresponds to the finite variation of the momentum:

$$\overline{H}'' - \overline{H}' = \sum_{i=1}^{n} \overline{P}_i \,, \tag{16.7}$$

$\overline{H}''$ and $\overline{H}'$ are respectively the linear momentums after and before the collision.

### 16.2.2. The theorem of angular momentum

The theorem of angular momentum for a material point is:

$$\frac{d\overline{K}_O}{dt} = \sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i \tag{16.8}$$

where $\overline{F}_i$ are only the impulsive external forces. It can be deduced successively:

$$d\overline{K}_O = \sum_{i=1}^{n} \left( \overline{r}_i \times \overline{F}_i \right) dt \Rightarrow \int_{t_1}^{t_2} d\overline{K}_O = \int_{t_1}^{t_2} \left[ \sum_{i=1}^{n} \left( \overline{r}_i \times \overline{F}_i \right) dt \right] = \sum_{i=1}^{n} \overline{r}_i \times \left( \int_{t_1}^{t_2} \overline{F}_i \right) = \sum_{i=1}^{n} \overline{r}_i \times \overline{P}_i \tag{16.9}$$

Due to the sudden change of the angular momentum, it can be written:

$$\overline{K}''_O - \overline{K}'_O = \sum_{i=1}^{n} \overline{r}_i \times \overline{P}_i \,, \tag{16.10}$$

$\overline{K}''_O$ and $\overline{K}'_O$ being respectively the angular moments after and before the collision.

### 16.2.3. The theorem of energy

The theorem of linear momentum for a particle of mass $m_i$ which simultaneously is submitted to percussion from the surrounding medium and from $n$ other particles, has the form:

$$m_i \overline{v}_i'' - m_i v_i' = \overline{P}_i + \sum_{j=1}^{n} \overline{P}_{ij} , \qquad (16.11)$$

in which it is assumed $\overline{P}_{ii} = \overline{0}$. Multiplying this equation by $\overline{v}_i''$, one gets:

$$m_i \left(\overline{v}_i''\right)^2 - m_i \overline{v}_i' \cdot \overline{v}_i'' = \overline{P}_i \cdot \overline{v}_i'' + \sum_{j=1}^{n} \overline{P}_{ij} \cdot \overline{v}_i'' . \qquad (16.12)$$

The left side of this equation can be cast into the form:

$$\frac{1}{2} m_i \left(\overline{v}_i''\right)^2 - \frac{1}{2} m_i \left(\overline{v}_i'\right)^2 + \frac{1}{2} m_i \left(\overline{v}_i'' - \overline{v}_i'\right)^2 = \overline{P}_i \cdot \overline{v}_i'' + \sum_{j=1}^{n} \overline{P}_{ij} \cdot \overline{v}_i'' \qquad (16.13)$$

For a system of n particles, the previous equation can be written for each particle. Adding these equations, it can be written:

$$\frac{1}{2}\sum_{i=1}^{n} m_i \left(\overline{v}_i''\right)^2 - \frac{1}{2}\sum_{i=1}^{n} m_i \left(\overline{v}_i'\right)^2 + \frac{1}{2}\sum_{i=1}^{n} m_i \left(\overline{v}_i'' - \overline{v}_i'\right)^2 = \sum_{i=1}^{n} \overline{P}_i \cdot \overline{v}_i'' + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{P}_{ij} \cdot \overline{v}_i'' \qquad (16.14)$$

or

$$T'' - T' + T_L = \sum_{i=1}^{n} \overline{P}_i \cdot \overline{v}_i'' + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{P}_{ij} \cdot \overline{v}_i'' \qquad (16.15)$$

where $T'$ and $T''$ are the kinetic energy before and respectively after collision and $T_L$ is the so called **kinetic energy of lost velocities**:

$$T_L = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\overline{v}_i'' - \overline{v}_i'\right)^2 \qquad (16.16)$$

If $\sum_{i=1}^{n} \overline{P}_i \cdot \overline{v}_i'' = 0$ and $\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{P}_{ij} \cdot \overline{v}_i'' = 0$, equation (16.15) becomes:

$$T' - T'' = T_L . \qquad (16.17)$$

This last relation is known as **Carnot's theorem** and proves that the energy of the system diminishes after percussion.

The condition $\sum_{i=1}^{n} \overline{P}_i \cdot \overline{v}_i'' = 0$ is accomplished if there are acting only percussions which are internal to the investigated system, so that $\overline{P}_i = \overline{0};\ i = 1,...,n$. The other condition $\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{P}_{ij} \cdot \overline{v}_i'' = 0$ can be accomplished in some particular cases: a) $\overline{v}'' = \overline{0}$:

perfectly plastic collision; b) $\overline{v}_i'' \perp \overline{P}_{ij}$: material points attached by inextensible strings.

### *16.3. Normal collision of two spheres*

The simplest example of impulsive motions is represented by two spheres which are moving along the line joining their mass centers.

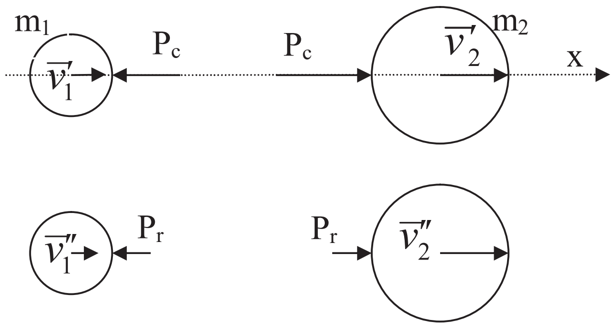



**Fig. 16.2 Normal collision of two spheres**

The axis *Ox* is along the line of centers, $m_1$, $m_2$ are the masses of the spheres; $\overline{v}_1', \overline{v}_2'$ and $\overline{v}_1'', \overline{v}_2''$ the velocities of the mass centers before and respectively after collision. $P_c$, $P_r$ are the magnitudes of the percussion during the compression and restitution periods respectively, k is the coefficient of restitution; *u* represents the common velocity of the centers at the end of the compression period which is also the beginning of the restitution period.
For the period of compression, the theorem of momentum for the two spheres can be written as:

$$\begin{aligned} m_1\left(u - v_1'\right) &= -P_c \\ m_2\left(u - v_2'\right) &= P_c \end{aligned}; \tag{16.18}$$

and for the period of restitution:

$$\begin{aligned} m_1\left(v_1'' - u\right) &= -P_r \\ m_2\left(v_2'' - u\right) &= P_r \end{aligned}. \tag{16.19}$$

The relation between percussions is:

$$P_r = kP_c. \tag{16.20}$$

By adding the four equations (16.18) and (16.19), the two percussions and the common velocity *u* are eliminated:

$$m_1v_1'' + m_2v_2'' = m_1v_1' + m_2v_2' \tag{16.21}$$

The ratios of the first equations of (16.18) and (16.19) as well as those of the second equations are both equal to the coefficient of restitution:

$$\frac{v_1'' - u}{u - v_1'} = \frac{v_2'' - u}{u - v_2'} = k\,, \tag{16.22}$$

or

$$v_1'' + kv_1' = v_2'' + kv_2' = u\left(k+1\right). \tag{16.23}$$

From the first equality of (16.23) it follows:

$$k = \frac{v_2'' - v_1''}{v_1' - v_2'}\,. \tag{16.24}$$

The velocities after collision can be deduced from (16.21) and (16.24):

$$\begin{aligned} v_1'' &= v_2'' - k\left(v_1' - v_2'\right) = \frac{m_1}{m_2}v_1' + v_2' - \frac{m_1}{m_2}v_1'' - k\left(v_1' - v_2'\right) = \left(\frac{m_1}{m_2} - k\right)v_1' + (1+k)v_2' - \frac{m_1}{m_2}v'' \\ v_2'' &= v_1'' + k\left(v_1' - v_2'\right) = v_1' + \frac{m_2}{m_1}v_2' - \frac{m_2}{m_1}v_2'' + k\left(v_1' - v_2'\right) = \left(\frac{m_2}{m_1} - k\right)v_2' + (1+k)v_1' - \frac{m_2}{m_1}v_2'' \end{aligned} \tag{16.25}$$

or

$$\begin{aligned} v_1'' &= \frac{m_1 - km_2}{m_1 + m_2}v_1' + \frac{\left(1+k\right)m_2}{m_1 + m_2}v_2' \\ v_2'' &= \frac{m_2 - km_1}{m_1 + m_2}v_2' + \frac{\left(1+k\right)m_1}{m_1 + m_2}v_1' \end{aligned} \tag{16.26}$$

The dissipated energy is

$$\begin{aligned} T' - T'' &= \frac{1}{2}m_1 v_1'^2 - \frac{1}{2}m_1 v_1''^2 + \frac{1}{2}m_2 v_2'^2 - \frac{1}{2}m_2 v_2''^2 \\ &= \frac{1}{2}m_1 m_2 \frac{2(1+k)m_1 + (1-k^2)m_2}{\left(m_1 + m_2\right)^2}v_1'^2 - \frac{1}{2}m_1 m_2 \frac{(1+k)^2 m_2}{\left(m_1 + m_2\right)^2}v_2'^2 - (1+k)m_1 m_2 \frac{m_1 - km_2}{\left(m_1 + m_2\right)^2}v_1' v_2' \\ &+ \frac{1}{2}m_1 m_2 \frac{2(1+k)m_2 + (1-k^2)m_1}{\left(m_1 + m_2\right)^2}v_2'^2 - \frac{1}{2}m_1 m_2 \frac{(1+k)^2 m_1}{\left(m_1 + m_2\right)^2}v_1'^2 - (1+k)m_1 m_2 \frac{m_2 - km_1}{\left(m_1 + m_2\right)^2}v_1' v_2' \\ &= \frac{m_1 m_2}{2\left(m_1 + m_2\right)^2}\left[\left(1-k^2\right)\left(m_1 + m_2\right)\left(v_1'^2 + v_1'^2\right) - 2\left(1-k^2\right)\left(m_1 + m_2\right)v_1' v_2'\right] \\ &= \frac{\left(1-k^2\right)m_1 m_2}{2\left(m_1 + m_2\right)}\left(v_1' - v_2'\right)^2 \end{aligned} \tag{16.27}$$

### 16.3.1. Perfectly elastic collision

If $k$=1 the velocities after collision are:

$$v_1'' = \frac{m_2}{m_1+m_2}\left[\left(\frac{m_1}{m_2}-1\right)v_1' + 2v_2'\right] = \frac{m_1-m_2}{m_1+m_2}v_1' + \frac{2m_2}{m_1+m_2}v_2'$$
$$v_2'' = \frac{m_1}{m_1+m_2}\left[\left(\frac{m_2}{m_1}-1\right)v_2' + 2v_1'\right] = \frac{m_2-m_1}{m_1+m_2}v_2' + \frac{2m_1}{m_1+m_2}v_1' \qquad (16.28)$$

If the two spheres have identical masses ($m_1=m_2$), then

$$v_1'' = v_2'; \quad v_2'' = v_1', \qquad (16.29)$$

which means that the spheres "exchange velocities". This case is particularly interesting because, in the kinetic theory of gases, the mathematical model presents the molecules as perfectly elastic spheres.

The dissipated kinetic energy from (16.27) is in this case ($k$=1):

$$T' - T'' = 0. \qquad (16.30)$$

### 16.3.2. Perfectly plastic collision

Considering $k = 0$ the velocities after collision are:

$$v_1'' = v_2'' = \frac{m_1 v_1' + m_2 v_2'}{m_1+m_2}, \qquad (16.31)$$

proving the two spheres have the same velocities after the collision.
The dissipated kinetic energy from (16.27) is in this case ($k$=0):

$$T' - T'' = \frac{m_1 m_2}{2\left(m_1+m_2\right)}\left(v_1' - v_2'\right)^2. \qquad (16.32)$$

Considering $v_2' = 0$, the rate of dissipated energy is:

$$\frac{T'-T''}{T'} = \frac{m_2}{m_1+m_2} = \frac{1}{1+m_1/m_2}. \qquad (16.33)$$

This formula has two practical applications:

- If a nail is to be inserted into a piece of wood, the rate of the dissipated kinetic energy must be very small during the collision with the hammer, transmitting a maximum of energy to the nail. Therefore the condition the mass $m_1$ of the hammer must be very large with respect to the mass $m_2$ of the nail ($m_1 \gg m_2$).
- On the contrary if the intention is to deform a piece of metal, the rate of the dissipated energy must be very large, since it must be transformed into energy of deformation. Therefore for a given mass $m_1$ of the hammer, the mass $m_2$ of the iron piece must be large, or increased by putting it on a large anvil.

## 16.4. Oblique collision of a sphere

Considering for example a spherical tennis ball of radius $r$ hitting the horizontal surface of ground with an oblique velocity $\vec{v}'$ of its mass center (the angle between the velocity vector and the vertical direction is α) and an angular velocity $\omega'$. If k is the restitution coefficient and $m$ and $J_C$ are respectively the mass and the moment of inertia about an axis passing through the center of the ball, it is to be determined the velocity $\vec{v}''$ and the angular velocity $\vec{\omega}''$ after the collision.

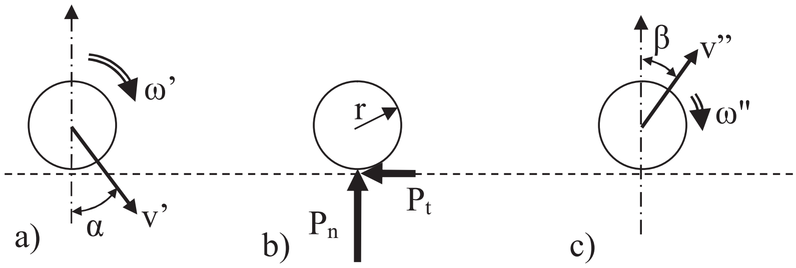



**Fig. 16.3 Oblique collision of a sphere against a planar surface**

Two percussions act on the tennis ball during the collision: one normal to the common tangent plane $P_n$ and another tangential $P_t$, which are the projections of the percussion as a vector, produced at the impact with the flat surface. Let be $\vec{v}''$, $\omega''$ and β, the velocity of the mass center, the angular velocity and respectively the angle between $\vec{v}''$ and the vertical direction after collision. The theorems of linear momentum and angular momentum are in this case:

$$\begin{cases} mv''\sin\beta - mv'\sin\alpha = -P_t \\ mv''\cos\beta - m(-v'\cos\alpha) = P_n \\ J_C\omega'' - J_C\omega' = rP_t \end{cases} \tag{16.34}$$

The relation between the percussions during the periods of restitution and compression is equivalent to a similar relation between the relative normal velocities (16.24) is in this case:

$$k = -\frac{v''_n}{v'_n} = -\frac{v''\cos\beta}{-v'\cos\alpha} \tag{16.35}$$

The equations (16.34) and (16.35) form a system of four equations with five unknowns: $v'', \beta, \omega'', P_n, P_t$. Another equation is thus necessary. Three hypotheses can be considered:

a) the surface of the ground is perfectly smooth. Then the fifth equation can be

$$P_t = 0\,. \tag{16.36}$$

In this case the equations become:

$$\begin{cases} v''\sin\beta = v'\sin\alpha \\ mv''\cos\beta - m(-v'\cos\alpha) = P_n \\ \omega'' = \omega' \\ v''\cos\beta = kv'\cos\alpha \end{cases}. \tag{16.37}$$

Consequently

$$v'' = v'\sqrt{\sin^2\alpha + k^2\cos^2\beta}; \quad \tan\beta = \frac{1}{k}\tan\alpha \quad \text{and} \quad \omega'' = \omega'. \tag{16.38}$$

Note that if k = 1 (perfectly elastic collision):

$$v'' = v'; \quad \beta = \alpha. \tag{16.39}$$

Therefore the modulus of the velocity doesn't change and the angle of reflection is equal to the angle of incidence. This case is particularly interesting because in the theory of light, the collision of photons with a reflecting surface is considered to be perfectly elastic. It is to be noted also that the components of linear momentum and consequently the velocity components parallel to the common tangent plane on the sphere undergo no changes:

$$v''\sin\beta = v'\sin\alpha. \tag{16.40}$$

b) the surface of the ground is rough and the relative motion of the ball with respect to the ground at the end of the collision is a pure rolling (without sliding). Then, the instantaneous center of rotation is located at the point of contact of the ball with the ground at the end of the restitution period, and the fifth equation is:

$$v_I = v''\sin\beta - r\omega'' = 0. \tag{16.41}$$

Supposing that before the impact, the ball had no angular velocity $\omega' = 0$, the equations are:

$$\begin{cases} mv''\sin\beta - mv'\sin\alpha = -P_t \\ mv''\cos\beta - m(-v'\cos\alpha) = P_n \\ J_C\omega'' = rP_t \\ v''\cos\beta = kv'\cos\alpha \\ v''\sin\beta = r\omega'' \end{cases} \tag{16.42}$$

Consequently the solutions are:

$$v'' = v'\sqrt{\left(\frac{mr^2}{J_C + mr^2}\right)^2 \sin^2\alpha + k^2\cos^2\alpha}; \quad \tan\beta = \frac{mr^2}{k\left(J_C + mr^2\right)}\tan\alpha$$
$$\omega'' = \frac{mrv'\sin\alpha}{J_C + mr^2} \tag{16.43}$$

Note that after impact, the ball will rotate about its center ($\omega'' \neq 0$) even if the initial motion of the ball had no such angular velocity $\omega' = 0$.

c) The surfaces in contact are rough, such that the tangent percussion is related to the normal percussion, by the dry sliding friction law of Coulomb:

$$P_t = \mu P_n, \tag{16.44}$$

in which $\mu$ is the sliding friction coefficient. This is the fifth equation, of the system:

$$\begin{cases} mv''\sin\beta - mv'\sin\alpha = -P_t \\ mv''\cos\beta - m(-v'\cos\alpha) = P_n \\ J_C\omega'' = rP_t \\ v''\cos\beta = kv'\cos\alpha \\ P_t = \mu P_n \end{cases} \tag{16.45}$$

From the first two and the fourth equations, it follows:

$$v''\cos\beta(\tan\beta + \mu) = kv'(\tan\beta + \mu)\cos\alpha = v'(\sin\alpha - \mu\cos\alpha) \tag{16.46}$$

from which:

$$\tan\beta = \frac{\tan\alpha - (1+k)\mu}{k}. \tag{16.47}$$

From (16.46):

$$v'' = v'\sqrt{k^2 + (\tan\alpha - (1+k)\mu)^2}\cos\alpha \tag{16.48}$$

From the third and first equations of (16.45) and (16.47) it follows:

$$\omega'' = \frac{rm}{J_C}v'\cos\alpha(\tan\alpha - k\tan\beta) = (1+k)\frac{\mu rm}{J_C}v'\cos\alpha. \tag{16.49}$$

As it can be seen, the percussion phenomenon is complex and more detailed information or hypotheses are needed for an accurate evaluation of the mechanical parameters after the impact.

## *16.5. Impacting a rigid body rotating about a fixed axis*

A rigid body is rotating about a fixed axis Oz (Fig. 16.4) and a movable Cartesian frame Oxyz attached to the body, such that the mass center C is in the Oxz plane. The points $O_1$ and $O_2$ are spherical joints on the Oz axis ($O_1O_2 = h$). The body is defined by its weight $\overline{G} = M\overline{g}$ with M the mass of the body $J_x$, $J_y$, $J_z$, $J_{xy}$, $J_{yz}$ , $J_{xz}$, the moments of inertia and the products of inertia of the body with respect to the Oxyz frame, and $C(\xi,0,\zeta)$ the coordinates of the mass center. Denoting by $\omega'$ and $\omega''$ the angular velocities before the collision and after it and projecting the given percussion $\overline{P}=P_x\overline{i}+P_y\overline{j}+P_z\overline{k}$ applied on a point of coordinates x, y, z of the body. The projections of the reaction percussions $\overline{P}_1$ and $\overline{P}_2$ in the spherical joints $O_1$ and $O_2$ are denoted by $P_{1x}$, $P_{1y}$, $P_{1z}$ and $P_{2x}$, $P_{2y}$, $P_{2z}$ respectively.

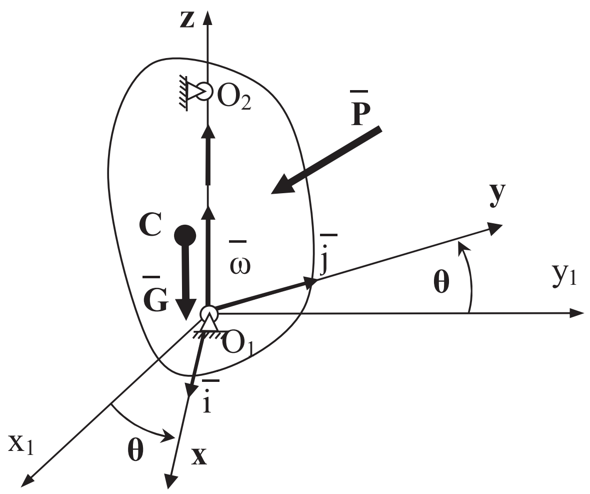



**Fig. 16.4 A percussion applied to a rigid body with a fixed axis**

The linear momentum and angular momentum expressions are for arbitrary ω:

$$\overline{H}=M\overline{v}=M\overline{\omega}\times\overline{r}_C=M\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 0 & 0 & \omega\\ \xi & 0 & \zeta\end{vmatrix}=M\omega\xi\overline{j}$$

$$[K_o]=\begin{bmatrix}J_x & -J_{xy} & -J_{xz}\\ -J_{yx} & J_y & -J_{yz}\\ -J_{zx} & -J_{zy} & J_z\end{bmatrix}\begin{bmatrix}0\\0\\\omega\end{bmatrix}=\begin{bmatrix}-J_{xz}\omega\\ -J_{yz}\omega\\ J_z\omega\end{bmatrix}\Rightarrow\overline{K}_o=-J_{xz}\omega\overline{i}-J_{yz}\omega\overline{j}+J_z\omega\overline{k} \tag{16.50}$$

The moments of the percussions are:

$$\overline{M}_o(\overline{P})=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ x & y & z\\ P_x & P_y & P_z\end{vmatrix}=(yP_z-zP_y)\overline{i}+(zP_x-xP_z)\overline{j}+(xP_y-yP_x)\overline{k}$$

$$\overline{M}_o(\overline{P}_2)=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 0 & 0 & h\\ P_{2x} & P_{2y} & P_{2z}\end{vmatrix}=hP_{2y}\overline{i}+hP_{2x}\overline{j} \tag{16.51}$$

The theorems of the linear and of the angular momentum can be written in this case:

$$\begin{aligned} \overline{H}'' - \overline{H}' &= \overline{P} + \overline{P}_1 + \overline{P}_2 \\ \overline{K}''_o - \overline{K}'_o &= \overline{r} \times \overline{P} + \overline{OO_1} \times \overline{P}_2 \end{aligned} \tag{16.52}$$

The six projections of these equations are:

$$\begin{aligned} &0 = P_x + P_{1x} + P_{2x} \\ &M\xi(\omega'' - \omega') = P_y + P_{1y} + P_{2y} \\ &0 = P_z + P_{1z} + P_{2z} \\ &-J_{xz}(\omega'' - \omega') = yP_z - zP_y + hP_{2y} \\ &-J_{yz}(\omega'' - \omega') = zP_x - xP_z + hP_{2x} \\ &J_z(\omega'' - \omega') = xP_y - yP_x \end{aligned} \tag{16.53}$$

There are six equations and seven unknowns. The percussions $P_{1z}$ and $P_{2z}$ cannot be determined because there is only one equation containing these unknowns. This situation has been already encountered for this constraint and can be easily solved if one spherical hinge is replaced by a cylindrical one, keeping the same mobility of the rigid body. If the spherical joint 2 is replaced by a cylindrical hinge then $P_{2z} = 0$ and $P_{1z} = -P_z$. The angular velocity after percussion can be obtained from the last equation:

$$\omega'' = \omega' + \frac{xP_y - yP_x}{J_z}. \tag{16.54}$$

### 16.5.1. Center of percussions

It is very important from the practical point of view to determine the conditions for which the reaction percussions $\overline{P}_1 = \overline{0}$ and $\overline{P}_2 = \overline{0}$. The equations (16.53) become:

$$\begin{aligned} &0 = P_x & &-J_{xz}(\omega'' - \omega') = yP_z - zP_y \\ &M\xi(\omega'' - \omega') = P_y & &-J_{yz}(\omega'' - \omega') = zP_x - xP_z \\ &0 = P_z & &J_z(\omega'' - \omega') = xP_y - yP_x \end{aligned} \tag{16.55}$$

These conditions can be expressed from the first and third equations, as:

$$P_x = 0; \quad P_z = 0. \tag{16.56}$$

This means that the percussion must be perpendicular to the plane defined by the axis of rotation and the mass center. Replacing these conditions in the remaining equations and in (16.54) these become:

$$M\xi(\omega''-\omega')=M\xi\frac{xP_y}{J_z}=P_y$$

$$-J_{xz}(\omega''-\omega')=-J_{xz}\frac{xP_y}{J_z}=-zP_y \quad , \qquad (16.57)$$

$$-J_{yz}(\omega''-\omega')=0$$

$$J_z(\omega''-\omega')=J_z\frac{xP_y}{J_z}=xP_y$$

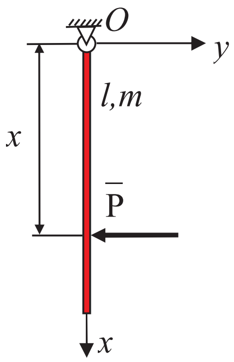
from which, the last one being an identity, three more conditions:

$$x=\frac{J_z}{M\xi};\qquad z=x\frac{J_{xz}}{J_z}=\frac{J_{xz}}{M\xi};\qquad J_{yz}=0 \qquad (16.58)$$

One consequence is that the given percussion P must cross the plane on which it is perpendicular in a point of coordinates x and z given by (16.58) and this point is called the **center of percussions** for the given body. The rigid body must have either the Oxy or the Oxz as symmetry planes so that $J_{yz}=0$.

Application 1

A rod of length *l* and mass *m* can rotate freely about a cylindrical hinge placed at one of its ends. Determine the center of percussions of the rod and the angular velocity $\omega''$ if $\omega'=0$.

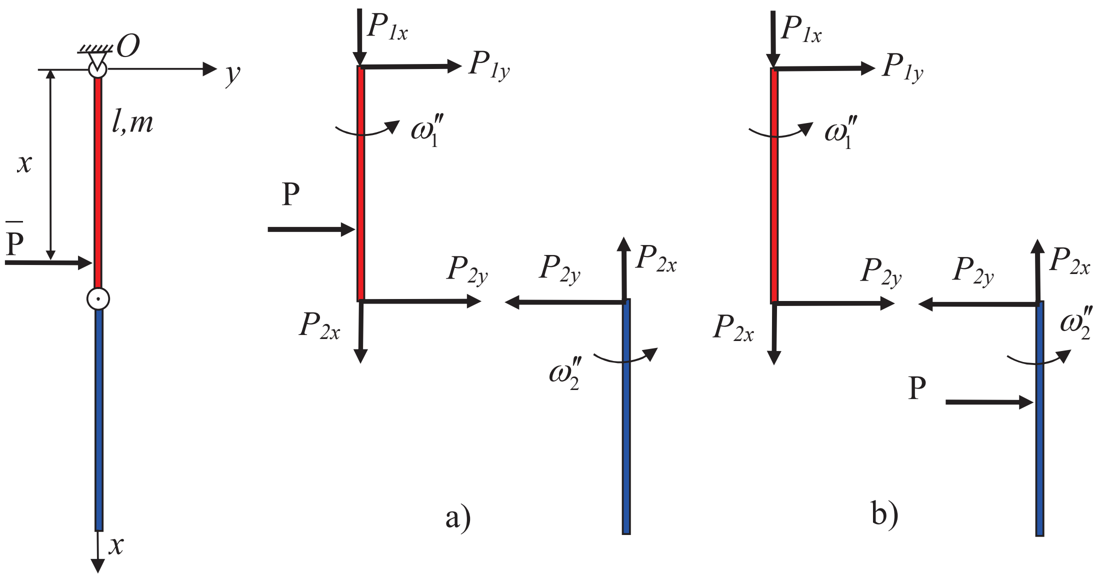



Due to the material line model $J_{yz}=\int_0^l yz\rho A dx=0$, in which ρ is the mass density and A the area of the cross section. The percussion must be perpendicular on the rod, which is situated in the plane defined by its mass center and the Oz axis. The only parameter to be determined from (16.58) is

$x=\frac{J_z}{M\xi}=\frac{ml^2}{3m\frac{l}{2}}=\frac{2}{3}l$. From (16.54) $\omega''=\frac{xP_y}{J_z}=\frac{2l}{3\frac{ml^2}{3}}P=\frac{2P}{ml}$

Application 2

Two rods are connected by a hinge and one rod is attached to a fixed hinge. The rods have identical lengths *l* and masses *m*. Determine their angular velocities after the application of a percussion P normal to the rod, at a distance x from the fixed hinge. The rods are at rest before the percussion.




a) If $x < l$ the theorems of linear and angular momentum can be written for each of the two rods (see figure a):

$$0 = P_{1x} + P_{2x} \qquad\qquad 0 = -P_{2x}$$

$$m\frac{l}{2}\omega_1'' - 0 = P + P_{1y} + P_{2y}\,; \qquad m\left(l\omega_1'' + \frac{l}{2}\omega_2''\right) - 0 = -P_{2y}$$

$$m\frac{l^2}{3}\omega_1'' - 0 = xP + lP_{2y} \qquad m\frac{l^2}{12}\omega_2'' - 0 = \frac{l}{2}P_{2y}$$

The solutions are obtained from the two last equations of the two groups:

$$P_{2y} = \frac{ml}{6}\omega_2'', \quad \omega_1'' = \frac{12}{7}\frac{Px}{ml^2}; \qquad \omega_2'' = -\frac{18}{7}\frac{Px}{ml^2}.$$

It follows that if the percussion is applied to the first rod, the first rod rotates anti-clockwise and the second rod rotates faster and in the opposite sense. The reaction percussion is $P_{1y} = m\frac{l}{2}\omega_1'' - P - P_{2y} = P\left(\frac{9x}{7l} - 1\right)$ and the percussion center for the first hinge is at $x = \frac{7}{9}l$.

b) If $x>l$ the percussion is applied on the second rod. The theorems of linear and angular momentum can be written for each of the two rods (see figure b):

$$0 = P_{1x} + P_{2x} \qquad\qquad 0 = -P_{2x}$$

$$m\frac{l}{2}\omega_1'' - 0 = P_{1y} + P_{2y}\,; \qquad m\left(l\omega_1'' + \frac{l}{2}\omega_2''\right) - 0 = P - P_{2y}$$

$$m\frac{l^2}{3}\omega_1'' - 0 = lP_{2y} \qquad m\frac{l^2}{12}\omega_2'' - 0 = \frac{l}{2}P_{2y} + P\left(x - \frac{3l}{2}\right)$$

The solutions are obtained from the last two equations of the two groups:

$$P_{2y} = \frac{ml}{3}\omega_1'' = \frac{P}{4} - \frac{\omega_2''}{8}ml\,;\ \omega_1'' + \frac{3}{8}\omega_2'' = \frac{3}{4}\frac{P}{ml}\,;$$

$$\omega_1'' = \frac{6P}{7ml^2}(5l - 3x)\,;\quad \omega_2'' = \frac{6P}{7ml^2}(8x - 11l)\,.$$

The next table summarizes the signs of the angular velocities, the positive sign corresponding to anticlockwise rotations.

| $x$ | $l$ | $11/8l$ | $5/3l$ | $2l$ |
|---|---|---|---|---|
| $\omega_1$ | + | + | 0 | - |
| $\omega_2$ | - | 0 | + | + |

The reaction percussions are: $P_{2y} = m\frac{l}{3}\omega_1'' = \frac{2P}{7l}(5l - 3x)$ and $P_{1y} = \frac{P}{7l}(5l - 3x)$. It is interesting to note that the percussion center $x = \frac{5}{3}l$ cancels both reaction percussions.

# 17. DYNAMICS OF A MATERIAL POINT WITH VARIABLE MASS

## *17.1. Basic Equation of the Motion of a Material Point with Variable Mass*

The motion of a body of variable mass, such as a rocket, changes its mass because some particles leave the body or are brought to the body during the motion. In the example of the rocket, when the fuel burns, gases are expelled propelling thus the rocket by the reaction force. The mass of the rocket diminishes therefore, by the mass of the evacuated gases.

If $m$ is the mass of a material point at time $t$ and $v$ its velocity, it can be considered that a small particle of mass $\Delta m$ and velocity $u$ is attached to the material point. At the moment $t + \Delta t$, the velocity of the material point shall be $v + \Delta v$. Applying the theorem of linear momentum for impulsive motions, it can be written:

$$\bar{H}'' - \bar{H}' = (m + \Delta m)(\bar{v} + \Delta\bar{v}) - m\bar{v} - \Delta m\bar{u} = \bar{F}\Delta t\,, \qquad (17.1)$$

where $\bar{F}$ is the force acting on the material point during the collision. Dividing by $\Delta$t and passing to the limit, with $\Delta t \to 0$, it results:

$$m\frac{d\bar{v}}{dt} = \bar{F} + \frac{dm}{dt}(\bar{u} - \bar{v}). \qquad (17.2)$$

Replacing $\bar{a} = \dfrac{d\bar{v}}{dt}$ the acceleration of the material point and considering $\bar{u} - \bar{v} = \bar{v}_r$ the relative velocity of the small particle of mass $\Delta m$ with respect to velocity of the material point of mass $m$, then the equation (17.2) can be written:

$$m\bar{a} = \bar{F} + \frac{dm}{dt}\bar{v}_r\,. \qquad (17.3)$$

## *17.2. The Motion of a Rocket*

A rocket moves along a straight line which is taken as the Ox axis. It can be assumed that $\bar{F} = 0$, the sum of external forces.

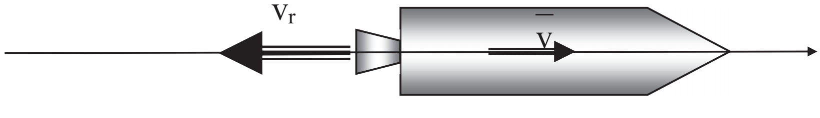



**Fig. 17.1 Motion of a rocket**

Since the Ox axis and $\overline{v}_r$ have opposite senses (Fig. 17.1), the equation (17.3) becomes:

$$m\ddot{x} = -\frac{dm}{dt}v_r. \tag{17.4}$$

Since $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dv}{dt}$, this equation can be written as:

$$m\frac{dv}{dt} = -\frac{dm}{dt}v_r \;\Rightarrow\; dv = -\frac{dm}{m}v_r. \tag{17.5}$$

Integrating this equation, it can be obtained:

$$v = -v_r \ln(m) + C. \tag{17.6}$$

If $v_0$ and $m_0$ are the velocity and the mass of the material point at $t = 0$, then

$$v_0 = -v_r \ln(m_0) + C \tag{17.7}$$

and the equation (17.6) can be written by eliminating the integration constant:

$$v = v_0 + v_r \ln\left(\frac{m_0}{m}\right). \tag{17.8}$$

If the rocket moves vertically upwards and the axis 0x is directed the same way, the equation (17.3) becomes:

$$m\ddot{x} = -\frac{dm}{dt}v_r - mg \tag{17.9}$$

and the solution becomes:

$$v = v_0 - gt + v_r \ln(m_0 / m). \tag{17.10}$$

If the imposed function of fuel combustion is linear, the mass of the rocket is:

$$m = m_0(1-\alpha t), \tag{17.11}$$

and the equation (17.9) can be written:

$$\frac{dx}{dt} = v_0 - gt + v_r \ln(1-\alpha t). \tag{17.12}$$

Integrating this equation, the equation of motion is:

$$x = x_0 + v_0 t - \frac{1}{2}gt^2 + \frac{v_r}{\alpha}\left[(1-\alpha t)\ln(1-\alpha t) + \alpha t\right] \tag{17.13}$$

# 18. PRINCIPLES OF ANALYTICAL MECHANICS

## *18.1. Preliminaries*

In Classical (Newtonian) Mechanics, the study of the motion of a material point is based on the fundamental equation $m\bar{a} = \bar{F}$ where $\bar{F}$ is the force acting on a free material point. This equation is not valid if the material point is constrained. In this case the geometrical conditions can be replaced by reactions according to the principle of constraints.

The problem is more complicated in the case of a system of material points because internal forces can play a role. In order to eliminate these forces, it is necessary to use the general theorems of dynamics. If the problem is complex, involving many particles, the elimination of the reactions is very difficult. Each problem of dynamics needs the use of a particular theorem and a dedicated method of solution.

Analytical Mechanics, founded by Lagrange, offers a general method permitting to eliminate the reactions and to write the equations of the motion in a general form. Analytical Mechanics is based on two principles:

- the principle of virtual work;
- the principle of d'Alembert.

Two problems have to be solved first: the problem of constraints and the problem of displacements.

## *18.2. Constraints*

A system of *n* material points $A_i$ (i = 1,..,n) is considered, having coordinates $A_i(x_i, y_i, z_i)$. If the constraints of the system can be expressed by *m* (*m* < *3n*) functional independent equations relating the coordinates:

$$F_j(x_1, y_1, z_1, ..., x_i, y_i, z_i, ..., x_n, y_n, z_n) = 0; \qquad j = 1, ..., m \qquad (18.1)$$

then the constraints are called **holonomic.** If all the constraints of the system are holonomic, the system of material points is called a **holonomic system**. The system of material points has in this case: $h = 3n - m$ degrees of freedom. It is possible to solve the system (18.1) with respect to selected *h* independent parameters $q_1$ ... $q_h$ in the form:

$$x_i = x_i(q_1, ..., q_h); \quad y_i = y_i(q_1, ..., q_h); \quad z_i = z_i(q_1, ..., q_h); \quad i = 1, ..., n \qquad (18.2)$$

The independent parameters $q_1$ ... $q_h$ are called **the Lagrange generalized coordinates**.

Example. Two material points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ are connected by a rigid rod. The constraint can be written:

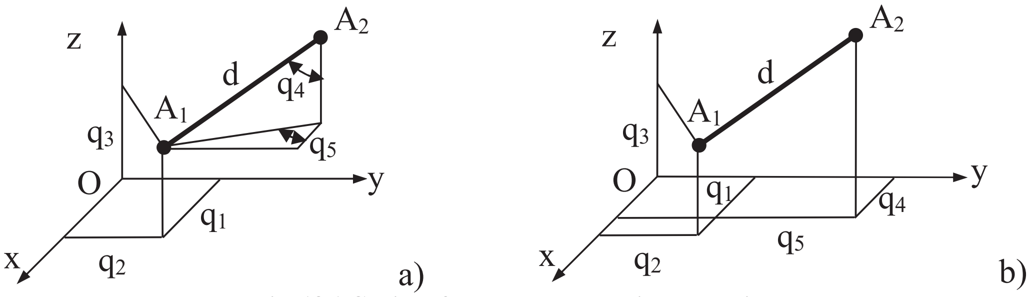


**Fig. 18.1 Choice of Lagrange generalized coordinates**

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-d^2=0, \tag{18.3}$$

in which $d$ is the constant distance between the two material points. The system has $3\cdot 2-1=5$ degrees of freedom. It can be proven that the relations:

$$\begin{aligned} &x_1=q_1;\quad y_1=q_2;\quad z_1=q_3;\\ &x_2=q_1+d\sin q_4\cos q_5\\ &y_2=q_2+d\sin q_4\sin q_5\\ &z_2=q_3+d\cos q_4 \end{aligned} \tag{18.4}$$

express the material coordinates $x_1, y_1, z_1, x_2, y_2, z_2$ with respect to five Lagrange generalized coordinates $q_1, q_2, q_3, q_4, q_5$ (Fig. 18.1a).

Note that the system of Lagrange generalized coordinates is not unique. It is possible to choose other Lagrange generalized coordinates $q_1, q_2, q_3, q_4, q_5$. For the same example (Fig. 18.1b):

$$\begin{aligned} &x_1=q_1;\quad y_1=q_2;\quad z_1=q_3;\\ &x_2=q_4\\ &y_2=q_5\\ &z_2=q_3+\sqrt{d^2-(q_4-q_1)^2-(q_5-q_2)^2} \end{aligned} \tag{18.5}$$

If the functions $F_j(x_1,y_1,z_1,...,x_i,y_i,z_i,...,x_n,y_n,z_n)=0;\ j=1,...,m$ are differentiable, then the equations (18.1) can be written in the differential form:

$$\frac{\partial F_j}{\partial x_1}dx_1+\frac{\partial F_j}{\partial y_1}dy_1+\frac{\partial F_j}{\partial z_1}dz_1+...\frac{\partial F_j}{\partial x_i}dx_i+\frac{\partial F_j}{\partial y_i}dy_i+\frac{\partial F_j}{\partial z_i}dz_i+...\frac{\partial F_j}{\partial x_n}dx_n+\frac{\partial F_j}{\partial y_n}dy_n+\frac{\partial F_j}{\partial z_n}dz_n=0 \tag{18.6}$$

$$j=1,...,m$$

This form proves that the $m$ constraints for holonomic system of material points can be expressed by relations between the natural coordinates $(x_i,y_i,z_i)$ as in

(18.1), or by *m* relations between the infinitesimal displacements $(dx_i, dy_i, dz_i)$ as in (18.6) and that these two forms are equivalent. Passing from the differential form to functions of coordinates is not always possible, because a differential form (or a system of differential forms) is not always integrable. For example the differential form:

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{18.7}$$

is not integrable for arbitrary functions P,Q and R, meaning that there is not always possible to find a function *f(x,y,z)* so that (18.7) is equivalent to:

$$f(x,y,z) = 0. \tag{18.8}$$

For this to be true, it is necessary that:

$$P(x,y,z) = \frac{\partial f(x,y,z)}{\partial x};\ Q(x,y,z) = \frac{\partial f(x,y,z)}{\partial y};\ R(x,y,z) = \frac{\partial f(x,y,z)}{\partial z} \tag{18.9}$$

or

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}; \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}. \tag{18.10}$$

If these conditions are not accomplished, then (18.7) is not integrable. The constraints are called **nonholonomic** if the constraints are expressed by differential non-integrable forms**.** If some of the constraints are nonholonomic, the system of material points is also called **nonholonomic**.

Example. The motion of a skate on ice without side-slip (Fig. 18.2). Let be *x* and *y* the coordinates of *C*, the central point of contact of the skate with the ice with respect to the Cartesian frame *Oxy* situated on the horizontal plane of the ice and by θ the angle between the skate and the *Ox* - axis.

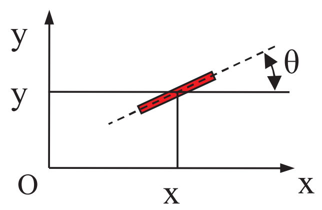



**Fig. 18.2 A skate moving on ice**

The condition of motion "without side - slip" imposes the relation between the components of the velocity for the central point (*x,y*) of the skate:

$$\frac{\dot{y}}{\dot{x}} = \tan\theta \quad \Rightarrow \quad dy - dx\tan\theta = 0. \tag{18.11}$$

This equation is non - integrable. Indeed, it has the general form: $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$, where:

$$P(x,y,z) = \tan\theta; \quad Q(x,y,z) = 1; \; R(x,y,z) = 0. \tag{18.12}$$

It follows:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0; \quad \frac{\partial Q}{\partial \theta} = \frac{\partial R}{\partial y} = 0; \text{ but } \quad \frac{\partial R}{\partial x} = 0 \neq \frac{\partial P}{\partial \theta} = -\frac{1}{\cos^2\theta} \tag{18.13}$$

Consequently no function $f(x,y,z) = 0$ can be obtained in this case. Therefore the parameters (*x, y, θ*) are independent, but the infinitesimal displacements (*dx, dy*) are not independent.

Considering the system of differential equations (18.6), it can be remarked that if the *m* relations are functionally independent the rank of the matrix :

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial z_1} & \ldots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_n} & \frac{\partial F_1}{\partial z_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial z_1} & \ldots & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_n} & \frac{\partial F_m}{\partial z_n} \end{bmatrix} \tag{18.14}$$

will be $r = m$.

If by replacing in (18.14) the natural coordinates $(x_i, y_i, z_i)$ ($i = 1,..,n$) by the Lagrange generalized coordinates $q_1$ ... $q_h$ the rank of the matrix (18.14) remains $r = m$, then the system of material points is holonomic. On the contrary if $r < m$, then the system of material points is called **critical**.

Example. Considering a system of two material points $A(x_1,y_1)$ and $B(x_2,y_2)$ situated on a circle of radius *r* and kept at all times at the extremities of a diameter (Fig. 18.3).

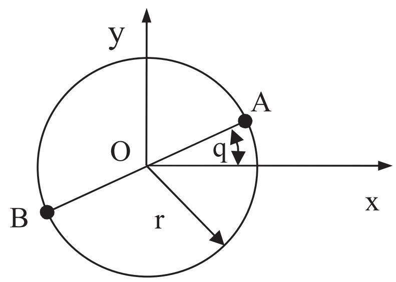



**Fig. 18.3 Two points linked by a diameter of a circle**

The relations expressing the constraints are:

$$\begin{aligned} &x_1^2 + y_1^2 - r^2 = 0; \quad x_2^2 + y_2^2 - r^2 = 0; \\ &(x_2 - x_1)^2 + (y_2 - y_1)^2 - 4r^2 = 0 \end{aligned} \tag{18.15}$$

These relations are functionally independent because the functional matrix:

$$\begin{bmatrix} 2x_1 & 2y_1 & 0 & 0 \\ 0 & 0 & 2x_2 & 2y_2 \\ -2(x_2 - x_1) & -2(y_2 - y_1) & 2(x_2 - x_1) & 2(y_2 - y_1) \end{bmatrix} \tag{18.16}$$

has the rank r = 3 since there is at least one non null determinant of order 3. However, by replacing the natural coordinates $x_1$, $y_1$, $x_2$, $y_2$ by their expressions with respect to the Lagrange generalized coordinate $q$ (Fig. 18.3):

$$\begin{aligned} x_1 &= r\cos q; \quad y_1 = r\sin q; \\ x_2 &= -r\cos q; \quad y_2 = -r\sin q; \end{aligned} \tag{18.17}$$

then the rank of the matrix (18.16) is $r = 2$. The number of degrees of freedom in infinitesimal displacements is 2, greater than the number of degrees of freedom in finite displacements.
The following table summarizes the discussion of the problem of constraints.

| The system | Number of degrees of freedom | |
|---|---|---|
| | Finite displacements | Infinitesimal displacements |
| Holonomic | h | $h_h = h$ |
| Nonholomic | h | $h_n < h$ |
| Critical | h | $h_c > h$ |

If the functions $F_j(x_1, y_1, z_1, ..., x_i, y_i, z_i, ..., x_n, y_n, z_n) = 0; \quad j = 1, ..., m$ do not depend on the time t, the constraints are **scleronomic** and the system of material points is called **scleronomous**.

Example. A simple (mathematical) pendulum is a material point attached by an inextensible string to a fixed point (Fig. 18.4a). The constraint is scleronomic, the function:

$$F(x, y) = \sqrt{x^2 + y^2} - l^2 = 0, \tag{18.18}$$

does not depend explicitly on time.

If the functions $F_j(x_1, y_1, z_1, ..., x_i, y_i, z_i, ..., x_n, y_n, z_n, t) = 0; \quad j = 1, ..., m$ depend explicitly on the time $t$, the constraints are called **rheonomic** and the system of material points is called **rheonomous**.

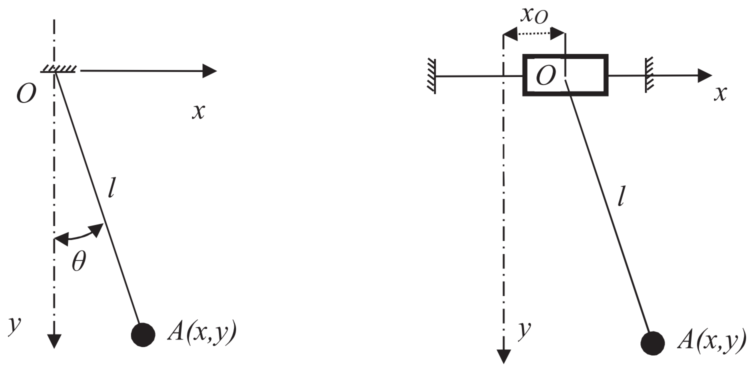



**Fig. 18.4 A scleronomous mathematical pendulum (a) and a rheonomous one (b).**

Example. A simple pendulum is attached by an inextensible string to a moving slider (Fig. 18.4b). If the slider moves following the time function $x_O = x_a \sin \omega t$ the constraint is rheonomic, since the function:

$$F(x, y) = \sqrt{(x - x_a \sin \omega t)^2 + y^2} - l^2 = 0, \tag{18.19}$$

is depending explicitly on time.

### *18.3. Displacements*

A rheonomic constraint is considered in the following, for example a movable or deformable surface (Fig. 18.5) at the moment of time *t* and at *t* + *dt* and a point A on this surface.

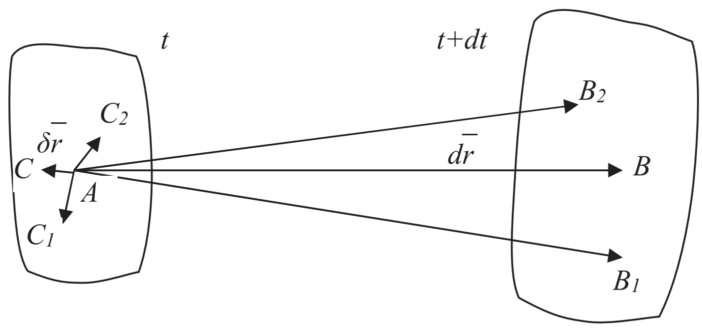



**Fig. 18.5 Real, possible and virtual displacements**

Supposing that the displacement of this point during the interval of time (*t, t+dt*) under the action of the force F applied onto the material point is $\overline{AB} = d\bar{r}$. The infinitesimal displacement $d\bar{r}$ is called **real displacement**.

If other forces would have acted on the material point, its displacements could have been $AB_1$ , $AB_2$ , ... . These displacements are called **possible displacements**. Obviously the real displacement is one of the possible ones.
Suppose that the surface is immobilized. Then the displacements of the material point can be $AC_1$, $AC_2$, ... . Let be $\delta\bar{r}$ such a displacement. It is called a **virtual displacement**.
If $f(x,y,z)=0$ is the equation of the surface, then the real displacement $d\bar{r}(dx,dy,dz)$ and the virtual displacements $\delta\bar{r}(\delta x,\delta y,\delta z)$ verify the following differential conditions:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt = 0 \\
\delta f &= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = 0
\end{aligned}
\tag{18.20}
$$

The virtual displacements have the following features:

a) are infinitesimal;
b) are compatible with the constraints at time *t*;
c) are independent of the time t;
d) are arbitrary.

Examining the formulas (18.20) it can be concluded that the "*δ*" operator acts as a "*d*" (differential) operator with the only difference that "*δ*" does not act on the variable "time" (*t*).

### *18.4. Principle of virtual work*

A material point A is constrained to move on a smooth surface and a given force $\bar{F}$ is acting on it. The constraint can be replaced according to the principle of constraints by a normal reaction N. It can be written:

$$\bar{N}\cdot\delta\bar{r} = 0 \tag{18.21}$$

because $\delta\bar{r}$ is a vector situated in the tangent plane in A to the surface and obviously $\bar{N}\perp\delta\bar{r}$. Since $\bar{N}$ is a force and $\delta\bar{r}$ a virtual displacement, the relation(18.21) can be considered as the expression of a virtual work $\delta W$ of the reaction force $\bar{N}$. This relation can be generalized for the case of a system of material points with only smooth constraints. If $\bar{R}_i;\ i=1,...,n$ are the reaction forces, from which some can be null, then

$$\delta W = \sum_{i=1}^{n}\bar{R}_i\delta\bar{r}_i = 0\,. \tag{18.22}$$

This relation expresses the general form for the **Principle of virtual work**:
"For a system of material points with smooth constraints, the virtual work of the reactions is null".

A system of material points acted by the given forces $\overline{F}_i;\ i=1,...,n$ . The necessary and sufficient conditions for the equilibrium are:

$$\overline{F}_i + \overline{R}_i = 0; \quad i=1,...,n, \tag{18.23}$$

where $\overline{R}_i;\ i=1,...,n$ are the reactions.

Multiplying these relations by $\delta\overline{r}_i$ using the scalar product and adding afterwards the obtained equations, (18.23) becomes:

$$\sum_{i=1}^{n} \overline{F}_i \delta\overline{r}_i + \sum_{i=1}^{n} \overline{R}_i \delta\overline{r}_i = 0. \tag{18.24}$$

Using (18.22) for the reaction forces, it follows that:

$$\delta W = \sum_{i=1}^{n} \overline{F}_i \delta\overline{r}_i = 0. \tag{18.25}$$

This relation is a particular form for the principle of virtual work:
"If a system of material points with smooth constraints is acted by the given forces $\overline{F}_i;\ i=1,...,n$, then the necessary and the sufficient condition for the equilibrium is that for every virtual displacement $\delta\overline{r}_i;\ i=1,...,n$ the virtual work of the given forces should be null".

Example 1. Determine the angle $\alpha$ for the position of equilibrium of the bar AB of length $l$ and mass $m$ shown on Fig. 18.6. The distance $a$ is given.

The only given force acting on the bar AB is its weight G applied at its mass center. A fixed Cartesian frame Oxy is considered with the origin in the corner. The virtual work of G must be null:

$$\delta W = -G\delta y_c = 0 \Rightarrow \delta y_c = 0. \tag{18.26}$$

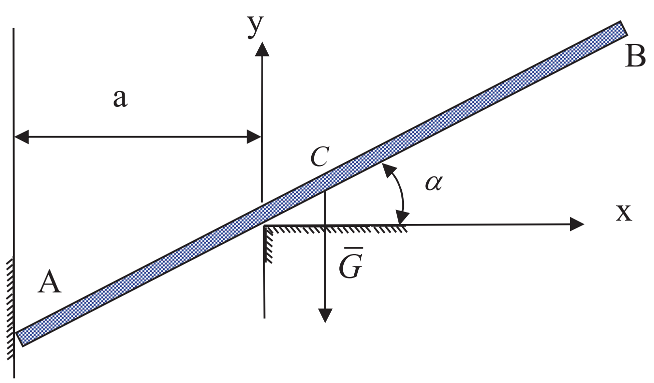



**Fig. 18.6 A bar at equilibrium simply supported at one end and against a fixed corner placed at a distance *a*.**

Since $y_c = \frac{1}{2}\sin\alpha - a\tan\alpha$, it follows

$$\delta y_c = \left(\frac{1}{2}\cos\alpha - \frac{a}{\cos^2\alpha}\right)\delta\alpha = 0 \Rightarrow \cos\alpha = \sqrt[3]{2a/l}\,. \tag{18.27}$$

Example 2. Determine the angles $\alpha_1$, $\alpha_2$ and $\alpha_3$ for the position of equilibrium of the system of light bars shown on Fig. 18.7. The bars have lengths $l_1$, $l_2$, $l_3$ and vertical forces are applied on the common hinges. A horizontal force F enforces equilibrium.

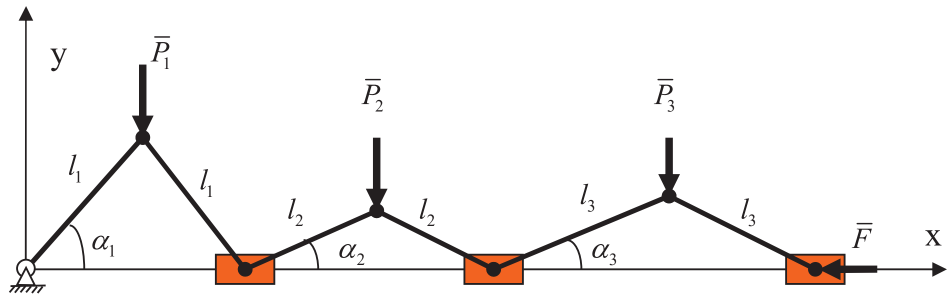



**Fig. 18.7 A system of rigid bars of negligible weight at equilibrium, having equal lengths by pairs.**

The given forces are $\overline{P}_1, \overline{P}_2, \overline{P}_3$ and $\overline{F}$. The fixed Cartesian frame Oxy is considered. The virtual work of these forces for every virtual displacement must be null. It follows:

$$\delta W = -P_1\delta y_A - P_2\delta y_B - P_3\delta y_C - F\delta x_D = 0\,. \tag{18.28}$$

The coordinates of the points where forces are applied are:

$$\begin{aligned} &y_A = l_1\sin\alpha_1;\; y_B = l_2\sin\alpha_2;\quad y_C = l_3\sin\alpha_3; \\ &x_D = l_1\cos\alpha_1 + l_2\cos\alpha_2 + l_3\cos\alpha_3 \end{aligned} \tag{18.29}$$

The components of the virtual displacements are:

$$\begin{aligned} &\delta y_A = l_1\cos\alpha_1\delta\alpha_1;\quad \delta y_B = l_2\cos\alpha_2\delta\alpha_2;\quad \delta y_C = l_3\cos\alpha_3\delta\alpha_3; \\ &\delta x_D = -l_1\sin\alpha_1\delta\alpha_1 - l_2\sin\alpha_2\delta\alpha_2 - l_3\sin\alpha_3\delta\alpha_3 \end{aligned} \tag{18.30}$$

Substituting these expressions in (18.28) one gets:

$$\begin{aligned} &\delta W = (-P_1l_1\cos\alpha_1 + Fl_1\sin\alpha_1)\delta\alpha_1 + (-P_2l_2\cos\alpha_2 + Fl_2\sin\alpha_2)\delta\alpha_2 + \\ &(-P_3l_3\cos\alpha_3 + Fl_3\sin\alpha_3)\delta\alpha_3 = 0 \end{aligned} \tag{18.31}$$

Since the condition (18.31) must be verified for every virtual displacements corresponding to the parameters $\delta\alpha_1, \delta\alpha_2, \delta\alpha_3$, three sets of values will be considered:

$$\begin{aligned} &\delta\alpha_1 \neq 0, \delta\alpha_2 = \delta\alpha_3 = 0; \\ &\delta\alpha_2 \neq 0, \delta\alpha_1 = \delta\alpha_3 = 0; \\ &\delta\alpha_3 \neq 0, \delta\alpha_1 = \delta\alpha_2 = 0. \end{aligned} \tag{18.32}$$

Three independent equations are thus obtained:

$$\begin{aligned} &-P_1 l_1 \cos\alpha_1 + F l_1 \sin\alpha_1 = 0; \\ &-P_2 l_2 \cos\alpha_2 + F l_2 \sin\alpha_2 = 0; \\ &-P_3 l_3 \cos\alpha_3 + F l_3 \sin\alpha_3 = 0; \end{aligned} \tag{18.33}$$

with the solutions:

$$\tan\alpha_1 = \frac{P_1}{F}; \quad \tan\alpha_2 = \frac{P_2}{F}; \quad \tan\alpha_3 = \frac{P_3}{F}. \tag{18.34}$$

## *18.5. Torricelli's Principle.*

A system of material points $A_i$ is considered, of masses $m_i (i = 1,...,n)$ acted only by their own gravity forces $m_i \overline{g}$. If the vertical axis Oz of the fixed Cartesian frame Oxyz has its sense upwards, then using the principle of virtual work, the condition of equilibrium of this system of forces can be written:

$$\sum_{i=1}^{n} -m_i \overline{g} \delta \overline{r_i} = -\sum_{i=1}^{n} m_i g \delta z_i = -g \sum_{i=1}^{n} m_i \delta z_i = -gM\,\delta\zeta = 0\,. \tag{18.35}$$

or in a simpler form:

$$\delta\zeta = 0\,. \tag{18.36}$$

in which $\zeta$ is the z coordinate of the mass center for the system of material points. It follows that "a system of gravity forces acting on a system of material points is in equilibrium if and only if the "altitude" $\zeta$ of the mass center of the system of material points has a stationary value (if $\zeta$ is a maximum, a minimum or simply stationary)". This represents the **Torricelli's principle**.

If $\zeta$ is a minimum, it can be proven that the equilibrium is **stable**.

If $\zeta$ is a maximum or simply stationary then the equilibrium is **unstable**.

An interesting application is the common catenary as a form of equilibrium for a uniform cable hanging freely under its own weight. If all the possible forms of a cable having the same length are considered, then the mass center of the common catenary has the minimum "altitude".

### *18.6. The principle of d'Alembert*

A material point is constrained to move on an inclined plane (Fig. 18.8a).

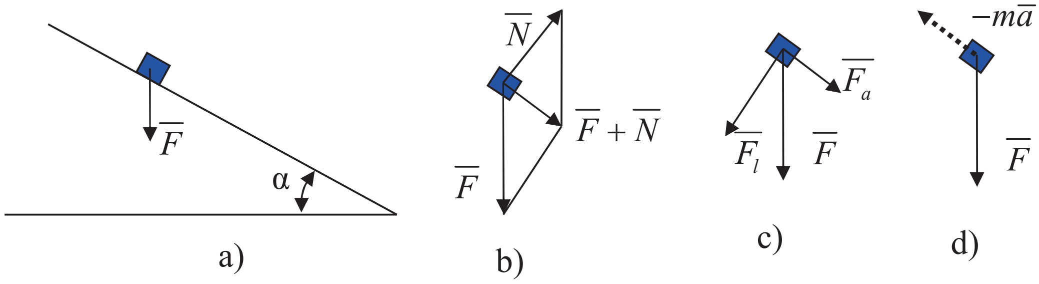



**Fig. 18.8 A material point moving on an inclined plane (a). Free body diagram (b), active and lost forces (c), pseudo-force (d).**

In classical mechanics the inclined plane is replaced by the reaction $\overline{N}$ (Fig. 18.8b) and the fundamental equation of dynamics is written as: $m\overline{a} = \overline{F} + \overline{N}$. It can be avoided the normal reaction $\overline{N}$ if the force $\overline{F}$ is projected as two components: $\overline{F_a}$ and $\overline{F_l}$. The component $\overline{F_a}$ is called **active force**. It contributes to the accelerated motion and is equal to *ma*. The component $\overline{F_l}$ has no effect on the frictionless motion of the material point. It is called **lost force**.

These considerations can be generalized for a system of material points.

Considering all the lost forces of a system of material points to be $\overline{F_{li}}$ (i = 1, ..., n), this system of forces has no effect on the motion of the system of material points. Since:

$$\begin{aligned} \overline{F_i} &= \overline{F_{ai}} + \overline{F_{li}} \\ \overline{F_{ai}} &= m_i \overline{a_i} \end{aligned} \tag{18.37}$$

it follows that:

$$\overline{F_{li}} = \overline{F_i} - \overline{F_{ai}} = \overline{F_i} - m_i \overline{a_i}\,. \tag{18.38}$$

The Principle of d'Alembert can be expressed in the II-nd form:

"By adding to the given forces $\overline{F_i}$ acting on the system of material points $A_i$ of masses $m_i$, the **fictitious forces** ($\tilde{F_i} = -m_i \overline{a_i}$), then the obtained system of forces is in a **fictitious equilibrium**".

The principle of d'Alembert considers the force $(-m_i \overline{a_i})$ acting on the material point $A_i$ (Fig. 18.8d). Obviously, in this case the force $(-m_i \overline{a_i})$ is fictitious or a **pseudo-force** and the equilibrium is fictitious or **pseudo-equilibrium**. The principle of d'Alembert reduces the investigation of the motion of material points, which is problem of dynamics, to a problem of statics.

Example. *Atwood's machine.*

At the ends of a string (inextensible, weightless) passing over a weightless pulley of radius $r$ are suspended two material points of masses $m_1$ and $m_2$ (Fig. 18.9a). Determine the acceleration $\bar{a}$ of the material point of mass $m_1$, the tension $\bar{T}$ in the string and the reaction $\bar{R}$ in the bearing of the pulley.

Two fictitious forces ($-m_1\bar{a}$ ) and ($-m_2\bar{a}$ ) are introduced over the real forces. The equations of equilibrium for the whole "rigid" system are:

$$\begin{aligned} &R_x = 0 \\ &R_y - m_1 g - m_2 g + m_1 a - m_2 a = 0 \ . \\ &(m_1 g - m_1 a)r - (m_2 g + m_1 a)r = 0 \end{aligned} \tag{18.39}$$

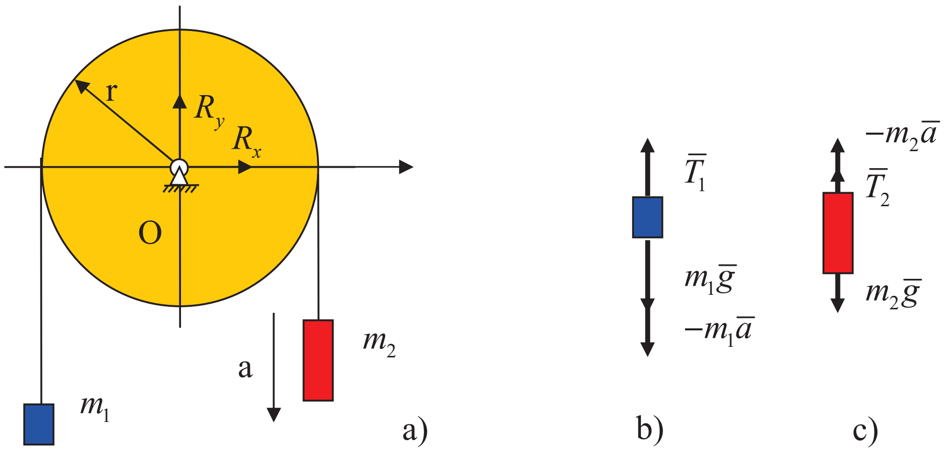



**Fig. 18.9 The Atwood machine. Model without moment of inertia (a). The masses with acting forces (b).**

The solutions are:

$$a = \frac{m_1 - m_2}{m_1 + m_2} g; \qquad R_x = 0; \qquad R_y = \frac{4m_1 m_2}{m_1 + m_2} g \tag{18.40}$$

Separating the material point $m_1$ (Fig. 18.9b), the equation of projection on a vertical axis is:

$$T_1 - m_1 g + m_1 a_1 = 0 \Rightarrow T_1 = \frac{2m_1 m_2}{m_1 + m_2} g \tag{18.41}$$

The same result is obtained if the material point of mass $m_2$ is isolated, because the pulley has no mass and thus no moment of inertia.

### 18.6.1. The Resultant vector and the Resultant moment of d'Alembert fictitious forces

The resultant vector and the resultant moment of d'Alembert fictitious forces can be written, by marking the fictitious forces and moments, instead of using the bar above sign "$\bar{}$ "for vectors, it will be used the wavy line sign "$\tilde{}$", as:

$$\tilde{R}=\sum_{i=1}^{n}(-m_i\bar{a}_i)=-\sum_{i=1}^{n}m_i\bar{a}_i=-\sum_{i=1}^{n}m_i\frac{d\bar{v}_i}{dt}=-\frac{d}{dt}\sum_{i=1}^{n}m_i\bar{v}_i=-\frac{d\bar{H}}{dt}$$

$$\tilde{M}_O=\sum_{i=1}^{n}\bar{r}_i\times(-\bar{m}_i\bar{v}_i)=-\sum_{i=1}^{n}\bar{r}_i\times\bar{m}_i\bar{v}_i=-\sum_{i=1}^{n}\bar{r}_i\times\bar{m}_i\frac{d\bar{v}_i}{dt} \tag{18.42}$$

$$=-\sum_{i=1}^{n}\frac{d}{dt}(\bar{r}_i\times\bar{m}_i\bar{v}_i)+\sum_{i=1}^{n}\frac{d\bar{r}_i}{dt}\times\bar{m}_i\bar{v}_i=-\sum_{i=1}^{n}\frac{d}{dt}(\bar{r}_i\times\bar{m}_i\bar{v}_i)=-\frac{d\bar{K}}{dt}$$

In the case of a rigid body having a *translation motion*:

$$\begin{aligned}\tilde{R}&=-M\bar{a}_c;\\ \tilde{M}_C&=-\bar{\rho}\times M\bar{a}_c\end{aligned} \tag{18.43}$$

where $M$ is the mass of the body, $\rho$ and $a$ are the position vector and the acceleration of the mass center respectively.

In the case of a rigid body having a *rotation motion* about the Oz axis:

$$\begin{aligned}\tilde{R}&=-M\bar{a}_c;\\ \tilde{M}_O&=(J_{xz}\varepsilon-J_{yz}\omega^2)\bar{i}+(J_{yz}\varepsilon+J_{xz}\omega^2)\bar{j}-J_z\varepsilon\bar{k}\end{aligned} \tag{18.44}$$

If $J_{xz}=J_{yz}=0$, then

$$\tilde{M}_O=-J_z\varepsilon\bar{k}\,. \tag{18.45}$$

It is important to note that in the fictitious equilibrium equation of moments, if the mechanical moments of inertia have been determined about the Oxyz frame, there will be not added the moment of the fictious resulting force $\tilde{R}=-M\bar{a}_c$. The reason is that the moments of elementary fictitious forces have already been included in $\tilde{M}_O$. On the contrary, if $\tilde{M}_O$ is determined using mechanical moments of inertia about the central frame (with the origin in the mass center), the moment of $\tilde{R}$ must be included in the equation of moments balance.

In the case of a *rigid body with a fixed point* O (axes $O_x, O_y, O_z$ are principal axes of inertia):

$$\tilde{R} = -M\bar{a}_c;$$

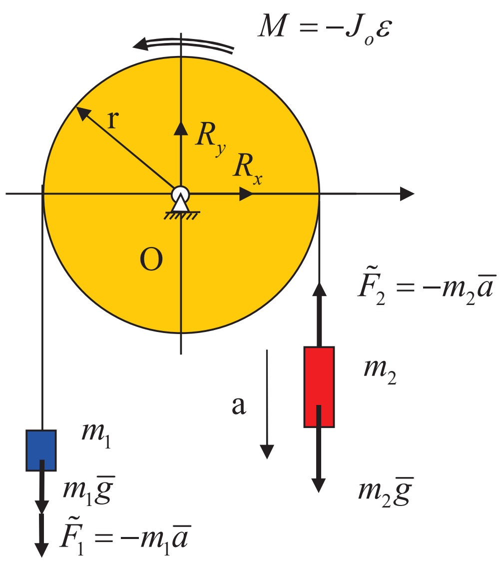
$$\tilde{M}_O = [-J_1\varepsilon_x - (J_3 - J_2)\omega_y\omega_z]\bar{i} + [-J_2\varepsilon_y - (J_1 - J_3)\omega_z\omega_x]\bar{j} + \qquad (18.46)$$

$$+[-J_3\varepsilon_z - (J_2 - J_1)\omega_x\omega_y]\bar{k}$$

Example 1. *Atwood's machine with moment of inertia.*
At the ends of a string (inextensible, weightless) passing over a pulley of mass $m$ and radius $r$ are suspended two material points of masses $m_1$ and $m_2$. Determine the acceleration $a$ of the material point of mass $m_1$.

Adding the fictitious forces ($\tilde{F}_1 - m_1\bar{a}$) and ($\tilde{F}_2 = -m_2\bar{a}$) and the fictitious couple ($\tilde{M}_0 = -J\bar{\varepsilon}$) to the real forces and moments, the system of three rigid bodies can be seen as a single rigid body for which equations of (fictitious) equilibrium can be written. The equation of moment balance with respect to the center of the pulley is:

$$(m_1 g - m_1 a)r - (m_2 g + m_2 a)r - J\varepsilon = 0. \qquad (18.47)$$




Replacing the mechanical moment of inertia about the center of the disc: $J_O = \frac{mr^2}{2}$ and the angular acceleration for a rotating disc of radius r and linear acceleration $a$ at its boundary, by $\varepsilon = \frac{a}{r}$, then the acceleration of the two masses results to be:

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{m}{2}} g \qquad (18.48)$$

The acceleration is smaller if the mechanical moment of inertia is taken into consideration.

**Fig. 18.10 Atwood machine. Model with moment of inertia.**

Example 2.
A homogeneous bar of length $l$ and mass M, rotates in a vertical plane about a horizontal axis passing through O, under the action of its own weight. At the initial moment $t = 0$, the bar is horizontal and at rest. Determine the angular acceleration $\varepsilon$, the angular velocity $\omega$, and the reaction R in O for a given angle $\theta$ (Fig. 18.11).

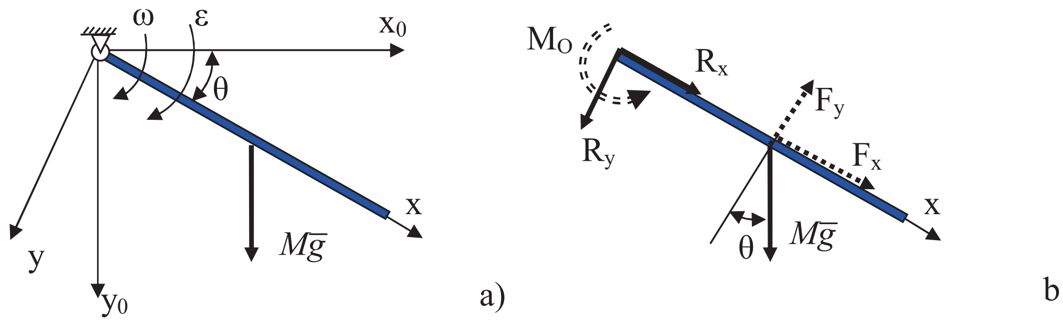



**Fig. 18.11 A bar rotating from initial rest (a). The free body diagram with applied pseudo-forces (b)**

Oxyz is a Cartesian frame attached to the bar and Ox coincides with the bar direction. $R_x$ and $R_y$ are the projections of the reaction $\bar{R}$ in the hinge. There will be added to the model, the fictitious forces $\tilde{F} = F_x\bar{i} + F_y\bar{j}$ and moment $\tilde{M} = -J_O\bar{\varepsilon}$ in the mass center of the bar:

$$\tilde{F}_x = -M\bar{a}_\nu = M\omega^2 \frac{l}{2}\bar{i}; \qquad \tilde{F}_y = -M\bar{a}_\tau = -M\varepsilon\frac{l}{2}\bar{j}$$
$$\tilde{M}_O = -J_O\bar{\varepsilon} = -M\frac{l^2}{3}\varepsilon\bar{k} \tag{18.49}$$

The equations of (fictitious) equilibrium on the Oxyz frame are :

$$R_X + Mg\sin\theta + M\omega^2\frac{l}{2} = 0; \qquad R_y + Mg\cos\theta - M\varepsilon\frac{l}{2} = 0$$
$$-J_O\varepsilon + Mg\frac{l}{2}\cos\theta = 0 \tag{18.50}$$

Remark: if instead of $J_O = \frac{Ml^2}{3}$ it had been used $J_C = M\frac{l^2}{12}$ and $\tilde{M}_C = -J_C\bar{\varepsilon} = -M\frac{l^2}{12}\varepsilon\bar{k}$, then the last equation form (18.50) would have been:

$$-J_C\varepsilon - M\varepsilon\frac{l}{2}\frac{l}{2} + Mg\frac{l}{2}\cos\theta = 0, \tag{18.51}$$

which leads to the same result. From the moment balance equations (18.50)(c) or (18.51), it follows:

$$\varepsilon = \frac{3}{2}\frac{g}{l}\cos\theta. \tag{18.52}$$

Multiplying by $\omega = \dot{\theta}$ $(\varepsilon = \ddot{\theta})$ and integrating it can be obtained successively:

$$\dot{\theta}\ddot{\theta} = \frac{3g}{2l}\dot{\theta}\cos\theta \;\Rightarrow\; \frac{\dot{\theta}^2}{2} = \frac{3g}{2l}\sin\theta + C \;\Rightarrow\; \omega^2 = \frac{3g}{2l}\sin\theta, \tag{18.53}$$

for the given initial condition, or

$$\omega = \pm\sqrt{\frac{3g}{l}\sin\theta}\,. \tag{18.54}$$

Replacing $\omega$ and $\varepsilon$ in the first two equations (18.50), it results:

$$R_x = -\frac{5Mg}{2}\sin\theta; \qquad R_y = -\frac{Mg}{4}\cos\theta\,; \tag{18.55}$$

the negative values indicating opposite orientations vs. those chosen on Fig. 18.11.

### *18.7. D'Alembert – Lagrange equation*

According to the principle of d'Alembert, the system of lost forces $\{\overline{F}_i - m_i\overline{a}_i\}$ has no effect on the system of material points, in other words it is in (fictitious) equilibrium. According to the principle of virtual work, if a system of forces is in equilibrium the virtual work of this system is equal to zero. Consequently, the following equations can be written:

$$\sum_{i=1}^{n}\left(\overline{F}_i - m_i\overline{a}_i\right)\delta\overline{r}_i = 0\,. \tag{18.56}$$

This represents the d'Alembert-Lagrange equation.

# 19. LAGRANGE EQUATIONS

## *19.1. Lagrange equations of the first kind*

A holonomic and rheonomic system of material points is considered. Since:

$$\overline{r}_i = \overline{r}_i(q_1,......,q_h,t)\,, \tag{19.1}$$

where $q_1...q_h$ are the Lagrange generalized coordinates, the virtual displacements:

$$\delta\overline{r}_i = \frac{\partial\overline{r}_i}{\partial q_1}\delta q_1 + ... + \frac{\partial\overline{r}_i}{\partial q_k}\delta q_k + ... + \frac{\partial\overline{r}_i}{\partial q_h}\delta q_h = \sum_{k=1}^{h}\frac{\partial\overline{r}_i}{\partial q_k}\delta q_k \tag{19.2}$$

and d'Alembert – Lagrange's equation (18.56) become:

$$\sum_{i=1}^{n}\left(\overline{F}_i - m_i\overline{a}_i\right)\delta\overline{r}_i = \sum_{i=1}^{n}\left(\overline{F}_i - m_i\overline{a}_i\right)\left(\sum_{k=1}^{h}\frac{\partial\overline{r}_i}{\partial q_k}\delta q_k\right) = \sum_{k=1}^{h}\left[\sum_{i=1}^{n}\frac{\partial\overline{r}_i}{\partial q_k}\left(\overline{F}_i - m_i\overline{a}_i\right)\right]\delta q_k = 0\,. \tag{19.3}$$

Since the system of material points is a holonomic one, the virtual displacements $\delta q_k$ are independent. If $\delta q_1 = 0,....,\delta q_{k-1} = 0,\ \delta q_k \neq 0,\ \delta q_{k+1} = 0,...,\delta q_h = 0$, it follows:

$$\sum_{i=1}^{n}\frac{\partial\overline{r}_i}{\partial q_k}\left(\overline{F}_i - m_i\overline{a}_i\right) = 0;\quad k = 1,...,h\,. \tag{19.4}$$

These equations can be written in the form:

$$\sum_{i=1}^{n}\frac{\partial\overline{r}_i}{\partial q_k}m_i\overline{a}_i = Q_k;\quad k = 1,...,h\,; \tag{19.5}$$

where

$$Q_k = \sum_{i=1}^{n}\frac{\partial\overline{r}_i}{\partial q_k}\overline{F}_i;\ \ k = 1,...,h\,; \tag{19.6}$$

are called **generalized forces**.

The equations (19.5) are known as **Lagrange equations of the first kind for holonomic systems**.

If the system of material points is non-holonomic, then $\delta q_1,....\delta q_h$ are not independent. Supposing there are *s* non-integrable differential relations (*s* < *h*):

$$\begin{aligned} &a_{ll}\delta q_l + ........ + a_{sh}\delta q_h = 0 \\ &............................... \\ &a_{sl}\delta q_1 + ........ + a_{sh}\delta q_h = 0 \end{aligned} \tag{19.7}$$

with:

$$a_{ij} = a_{ij}(q_1..........q_h,t)\,. \tag{19.8}$$

Applying Lagrange's method of multipliers instead of (19.3) it can be obtained the equation:

$$\sum_{k=1}^{h}\left[\sum_{i=1}^{n}(\overline{F}_i - m_i\overline{a}_1)\frac{\partial \overline{r}_i}{\partial q_k}\right]\delta q_k + \sum_{j=1}^{s}\lambda_j\left(\sum_{k=1}^{h}a_{jk}\delta q_k\right) = 0, \qquad (19.9)$$

where $\lambda_j (j = 1,.....,s)$ are unknown multipliers. The multipliers can be determined by imposing that $q_k (k = 1,.....,h)$ to be independent. The equation (19.9) can be written in the form:

$$\sum_{k=1}^{h}\left[\sum_{i=1}^{n}(\overline{F}_i - m_i\overline{a}_1)\frac{\partial \overline{r}_i}{\partial q_k} + \sum_{j=1}^{s}\lambda_j a_{jk}\right]\delta q_k = 0. \qquad (19.10)$$

The parameters $q_k$ correspond to independent virtual displacements so that (19.10) is equivalent to *h* equations:

$$\sum_{i=1}^{n}(\overline{F}_i - m_i\overline{a}_1)\frac{\partial \overline{r}_i}{\partial q_k} + \sum_{j=1}^{s}\lambda_j a_{jk} = 0; \quad k = 1,...,h. \qquad (19.11)$$

The equations (19.11) are known as **Lagrange equations of the first kind for non-holonomic systems**.

## *19.2. Lagrange equations of the second kind*

The expression $\sum_{i=1}^{n} m_i\overline{a}_i\frac{\partial \overline{r}_i}{\partial q_k}$ can be written in the form:

$$\sum_{i=1}^{n} m_i\overline{a}_i\frac{\partial \overline{r}_i}{\partial q_k} = \sum_{i=1}^{n} m_i\frac{d\overline{v}_i}{dt}\frac{\partial \overline{r}_i}{\partial q_k} = \frac{d}{dt}\left[\sum_{i=1}^{n} m_i\overline{v}_i\frac{\partial \overline{r}_i}{\partial q_k}\right] - \sum_{i=1}^{n} m_i\overline{v}_i\frac{d}{dt}\left(\frac{\partial \overline{r}_i}{\partial q_k}\right). \qquad (19.12)$$

Differentiating the position vector (19.1) with respect to time will be obtained the velocity:

$$\overline{v}_i = \frac{\partial \overline{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \overline{r}_i}{\partial q_2}\dot{q}_2 + ... + \frac{\partial \overline{r}_i}{\partial q_k}\dot{q}_k + ... + \frac{\partial \overline{r}_i}{\partial q_h}\dot{q}_h + \frac{\partial \overline{r}_i}{\partial t}. \qquad (19.13)$$

Considering $\overline{v_i}$ a vector function of the Lagrange generalized coordinates $q_1,....,q_h$ and of the Lagrange generalized velocities $\dot{q}_1, \dot{q}_h$ as all these *2h* parameters were independent, then:

$$\frac{\partial v_i}{\partial \dot{q}_k} = \frac{\partial \overline{r}_i}{\partial q_k} \qquad (19.14)$$

because $\overline{v}_i$ is a linear function of $\dot{q}_k$ and $\frac{\partial \overline{r}_i}{\partial q_k}$ is the "coefficient" of $\dot{q}_k$.

Calculating $\frac{d}{dt}\left(\frac{\partial \overline{r}_i}{\partial q_k}\right)$ and $\frac{\partial \overline{v}_i}{\partial q_k}$ one gets:

$$\frac{d}{dt}\left(\frac{\partial \overline{r}_i}{\partial q_k}\right)=\frac{\partial^2 \overline{r}_i}{\partial q_1 \partial q_k}\dot{q}_1+\frac{\partial^2 \overline{r}_i}{\partial q_2 \partial q_k}\dot{q}_2+...+\frac{\partial^2 \overline{r}_i}{\partial q_k \partial q_k}\dot{q}_k+...+\frac{\partial^2 \overline{r}_i}{\partial q_h \partial q_k}\dot{q}_h+\frac{\partial^2 \overline{r}_i}{\partial t \partial q_k}$$

$$\frac{\partial \overline{v}_i}{\partial q_k}=\frac{\partial^2 \overline{r}_i}{\partial q_k \partial q_1}\dot{q}_1+\frac{\partial^2 \overline{r}_i}{\partial q_k \partial q_2}\dot{q}_2+...+\frac{\partial^2 \overline{r}_i}{\partial q_k \partial q_k}\dot{q}_k+...+\frac{\partial^2 \overline{r}_i}{\partial q_k \partial q_h}\dot{q}_h+\frac{\partial^2 \overline{r}_i}{\partial q_k \partial t}=\frac{d}{dt}\left(\frac{\partial \overline{r}_i}{\partial q_k}\right) \tag{19.15}$$

It follows that:

$$\frac{d}{dt}\left(\frac{\partial \overline{r}_i}{\partial q_k}\right)=\frac{\partial \overline{v}_i}{\partial q_k}. \tag{19.16}$$

Replacing $\frac{\partial \overline{r}_i}{\partial q_k}$ and $\frac{d}{dt}\left(\frac{\partial \overline{r}_i}{\partial q_k}\right)$ by their expressions (19.14) and (19.16) it is obtained:

$$\frac{d}{dt}\left[\sum_{i=1}^{n} m_i \overline{v}_i \frac{\partial \overline{r}_i}{\partial q_k}\right]=\frac{d}{dt}\left[\sum_{i=1}^{n} m_i \overline{v}_i \frac{\partial \overline{v}_i}{\partial \dot{q}_k}\right]=\frac{d}{dt}\left[\sum_{i=1}^{n} m_i \frac{\partial}{\partial \dot{q}_k}\left(\frac{\overline{v}_i \cdot \overline{v}_i}{2}\right)\right]$$
$$=\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{n}\frac{m_i v_i^2}{2}\right)\right]=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)$$
$$\sum_{i=1}^{n} m_i \overline{v}_i \frac{d}{dt}\left(\frac{\partial \overline{r}_i}{\partial q_k}\right)=\sum_{i=1}^{n} m_i \overline{v}_i \frac{\partial \overline{v}_i}{\partial q_k}=\sum_{i=1}^{n} m_i \frac{\partial}{\partial q_k}\left(\frac{\overline{v}_i \cdot \overline{v}_i}{2}\right)$$
$$=\frac{\partial}{\partial q_k}\left(\sum_{i=1}^{n}\frac{m_i v_i^2}{2}\right)=\frac{\partial T}{\partial q_k} \tag{19.17}$$

where $T$ is the kinetic energy of the system of material points. Replacing these expressions in (19.12), it can be deduced:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k;\ \ k=1,...,h. \tag{19.18}$$

which represent the **Lagrange equations of the second kind for holonomic systems of material points**.

In a similar manner it can be deduced:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k+\sum_{j=1}^{s}\lambda_j a_{jk};\ \ k=1,...,h, \tag{19.19}$$

which represent the **Lagrange equations of the second kind for non-holonomic systems of material points**, to which must be added the $s$ equations (19.7).

Note : The generalized forces can be calculated using the formula (19.6). Another method is based on the expression of the virtual work of given forces:

$$\delta W=\sum_{i=1}^{n}\overline{F}_i\delta\overline{r}_i=\sum_{i=1}^{n}\overline{F}_i\left(\sum_{k=1}^{h}\frac{\partial\overline{r}_i}{\partial q_k}\delta q_k\right)=\sum_{k=1}^{h}\left(\sum_{i=1}^{n}\overline{F}_i\frac{\partial\overline{r}_i}{\partial q_k}\right)\delta q_k=\sum_{k=1}^{h}Q_k\delta q_k\,. \quad (19.20)$$

If $\delta q_1=0...,\delta q_{k-1}=0,\ \delta q_k\neq 0,\ \delta q_{k+1}=0,....,\delta q_h=0$, then can be obtained for each generalized coordinate:

$$\delta_k W=Q_k\delta q_k\,, \quad (19.21)$$

where $\delta_k W$ is the virtual work if the Lagrange generalized coordinate $q_k$ has a virtual variation $\delta q_k$. The generalized force $Q_k$ is the coefficient of $\delta q_k$ in the expression of $\delta_k W$.

### 19.2.1. Case of a conservative system of given forces

If all the given forces $\overline{F}_i$ are conservative, which means that their components $X_i$, $Y_i$, $Z_i$ can be obtained are the derivatives of so called **force functions**:

$$X_i=\frac{\partial U_i}{\partial x_i};\quad Y_i=\frac{\partial U_i}{\partial y_i};\quad Z_i=\frac{\partial U_i}{\partial z_i};\quad i=1,...,n\,; \quad (19.22)$$

then

$$\delta L=\sum_{i=1}^{n}\overline{F}_i\delta\overline{r}_i=\sum_{i=1}^{n}\left(X_i\delta x_i+Y_i\delta y_i+Z_i\delta z_i\right)=$$
$$\sum_{i=1}^{n}\left(\frac{\partial U_i}{\partial x_i}\delta x_i+\frac{\partial U_i}{\partial y_i}\delta y_i+\frac{\partial U_i}{\partial z_i}\delta z_i\right)=\sum_{i=1}^{n}\delta U_i=\delta\sum_{i=1}^{n}U_i=\delta U=\sum_{k=1}^{h}\frac{\partial U}{\partial q_k}\delta q_k \quad (19.23)$$

Comparing (19.23) to (19.20) it can be obtained:

$$Q_k=\frac{\partial U}{\partial q_k};\qquad k=1,....,h\,. \quad (19.24)$$

It follows that the Lagrange equations in this case are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=\frac{\partial U}{\partial q_k};\ \ k=1,...,h\,. \quad (19.25)$$

These are the **Lagrange equations of the second kind for holonomic systems of material points if all generalized forces come from force functions.** Denoting by:

$$L=T+U\,, \quad (19.26)$$

the function $L$ is called the **Lagrange kinetic potential**. Since the kinetic energy T is a function of $q_1,...,q_h,\dot{q}_1,...,\dot{q}_h$ and U is a function of $q_1,...,q_h,t$, it follows that:

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}; \quad \frac{\partial L}{\partial q_k} = \frac{\partial T}{\partial q_k} + \frac{\partial U}{\partial q_k} \tag{19.27}$$

Replacing these expressions in (19.18) another form of the Lagrange's equations for holonomic systems is obtained, using the kinetic potential:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k; \quad k = 1,...,h \tag{19.28}$$

and by replacing the expressions (19.27) in (19.19) another form of the Lagrange's equations for non-holonomic systems is obtained, using the kinetic potential

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k + \sum_{j=1}^{s} \lambda_j a_{jk}; \quad k = 1,...,h. \tag{19.29}$$

Example 1. Write the Lagrange equations for the system made of a slider of mass $m_1$ and a point of mass $m_3$, connected by a rod AB of length $l$ and mass $m_2$. The constraints are smooth (fig. 19.1).

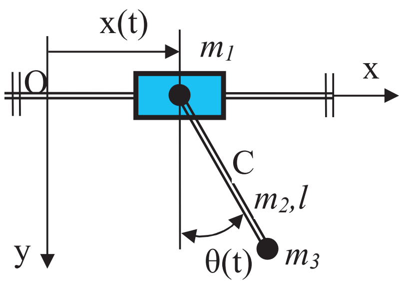



**Fig. 19.1 A mechanical system with two degrees of freedom. A linear parameter and an angular one.**

The system is scleronomous and holonomic. The two generalized coordinates are x(t) and θ(t). The energy of the slider is $T_1 = \frac{1}{2}m_1\dot{x}^2$. The mass center C of the rod has a velocity

$$\bar{v}_C = \left(\dot{x} + \frac{l}{2}\dot{\theta}\cos\theta\right)\bar{i} + \frac{l}{2}\dot{\theta}\sin\theta\bar{j} \tag{19.30}$$

and the energy of the rod is

$$T_2 = \frac{1}{2}m_2v_C^2 + \frac{1}{2}J_C\omega^2 = \frac{1}{2}m_2\left(\dot{x}^2 + l\dot{\theta}\dot{x}\cos\theta + \frac{l^2}{4}\dot{\theta}^2\right) + \frac{1}{2}\frac{m_2l^2}{12}\dot{\theta}^2$$
$$= \frac{1}{2}m_2\left(\dot{x}^2 + l\dot{\theta}\dot{x}\cos\theta + \frac{l^2}{3}\dot{\theta}^2\right) \tag{19.31}$$

The material point $m_3$ has a velocity

$$\overline{v}_3 = \left(\dot{x} + l\dot{\theta}\cos\theta\right)\overline{i} + l\dot{\theta}\sin\theta\overline{j} \tag{19.32}$$

and energy

$$T_3 = \frac{1}{2}m_3 v_3^2 = \frac{1}{2}m_3\left(\dot{x}^2 + 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2\right). \tag{19.33}$$

The kinetic energy of the whole system is:

$$E = \frac{1}{2}\left(m_1 + m_2 + m_3\right)\dot{x}^2 + \frac{1}{2}\left(m_2 + 2m_3\right)l\dot{\theta}\dot{x}\cos\theta + \frac{1}{2}\left(\frac{m_2}{3} + m_3\right)l^2\dot{\theta}^2. \tag{19.34}$$

The force function is in this case:

$$U = m_1 g 0 + m_2 g\frac{l}{2}\cos\theta + m_3 gl\cos\theta = \left(\frac{m_2}{2} + m_3\right)gl\cos\theta. \tag{19.35}$$

The required terms for the first Lagrange equation are:

$$\begin{aligned}
&\frac{\partial E}{\partial \dot{x}} = \left(m_1 + m_2 + m_3\right)\dot{x} + \frac{1}{2}\left(m_2 + 2m_3\right)l\dot{\theta}\cos\theta \\
&\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}}\right) = \left(m_1 + m_2 + m_3\right)\ddot{x} + \frac{1}{2}\left(m_2 + 2m_3\right)l\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \\
&\frac{\partial E}{\partial x} = 0; \quad \frac{\partial U}{\partial x} = 0;
\end{aligned} \tag{19.36}$$

and for the second one:

$$\begin{aligned}
&\frac{\partial E}{\partial \dot{\theta}} = \frac{1}{2}\left(m_2 + 2m_3\right)l\dot{x}\cos\theta + \left(\frac{m_2}{3} + m_3\right)l^2\dot{\theta} \\
&\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\theta}}\right) = \frac{1}{2}\left(m_2 + 2m_3\right)l\left(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta\right) + \left(\frac{m_2}{3} + m_3\right)l^2\ddot{\theta} \\
&\frac{\partial E}{\partial \theta} = -\frac{1}{2}\left(m_2 + 2m_3\right)l\dot{x}\sin\theta; \quad \frac{\partial U}{\partial \theta} = -\left(\frac{m_2}{2} + m_3\right)gl\sin\theta
\end{aligned} \tag{19.37}$$

Replacing all these terms in (19.25), the Lagrange equations are:

$$\begin{cases}
\left(m_1 + m_2 + m_3\right)\ddot{x} + \frac{1}{2}\left(m_2 + 2m_3\right)l\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = 0 \\
\frac{1}{2}\left(m_2 + 2m_3\right)l\left(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta\right) + \left(\frac{m_2}{3} + m_3\right)l^2\ddot{\theta} + \frac{1}{2}\left(m_2 + 2m_3\right)l\dot{x}\sin\theta = -\left(\frac{m_2}{2} + m_3\right)gl\sin\theta
\end{cases} \tag{19.38}$$

Example 2. Write the Lagrange equation for the motion of a heavy material point A of mass $m$ constrained to move on a bar which rotates uniformly about a vertical axis and study the motion (Fig. 19.2).

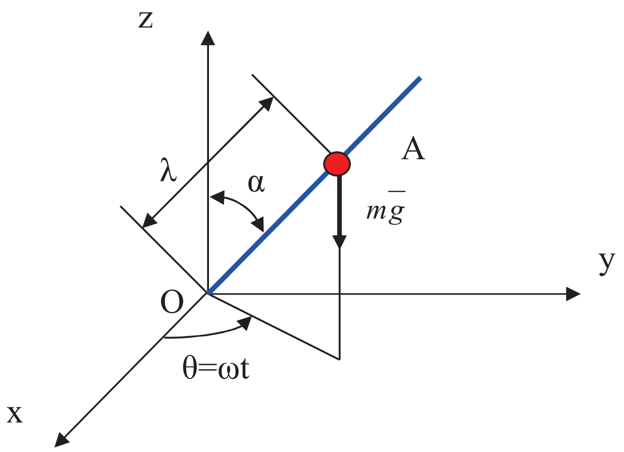



**Fig. 19.2 A point sliding freely on a bar rotating uniformly about the Oz axis**

This system is a rheonomic system, with the single degree of freedom. It can be chosen *OA=λ(t)* as Lagrange generalized coordinate. The natural coordinates of A are:

$$\mathrm{x} = \lambda \sin\alpha \ \cos\omega \mathrm{t};\ \mathrm{y} = \lambda \sin\alpha \ \sin\omega \mathrm{t}\ ;\ \mathrm{z} = \lambda \cos\alpha \tag{19.39}$$

The projections of the velocity on the axis of the fixed Cartesian frame Oxyz are:

$$\begin{aligned} \dot{x} &= \dot{\lambda}\sin\alpha\cos\omega t - \lambda\omega\sin\alpha\sin\omega t; \\ \dot{y} &= \dot{\lambda}\sin\alpha\sin\omega t + \lambda\omega\sin\alpha\cos\omega t \\ \dot{z} &= \dot{\lambda}\cos\alpha \end{aligned} \tag{19.40}$$

The expression of the kinetic energy of the material point A is:

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{m}{2}\left(\dot{\lambda}^2 + \lambda^2\omega^2\sin^2\alpha\right) \tag{19.41}$$

The force function U has the expression:

$$U = -m\ g\ z = -\ mg\lambda\ \cos\alpha \tag{19.42}$$

The kinetic potential L has the expression:

$$L = T + U = \frac{m}{2}\left(\dot{\lambda}^2 + \lambda^2\omega^2\sin^2\alpha\right) - \ mg\lambda\ \cos\alpha \tag{19.43}$$

It follows that:

$$\frac{\partial L}{\partial \dot{\lambda}} = m\dot{\lambda};\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = m\ddot{\lambda};\quad \frac{\partial L}{\partial \lambda} = m\lambda\omega^2\sin^2\alpha - mg\cos\alpha \tag{19.44}$$

The Lagrange equation is:

$$m\ddot{\lambda} - m\lambda\omega^2\sin^2\alpha + mg\cos\alpha = 0\,. \tag{19.45}$$

This differential equation can be written:

$$\ddot{\lambda} - \lambda\omega^2 \sin^2\alpha = -mg\cos\alpha . \quad (19.46)$$

It is a non-homogeneous linear differential equation. The solution is:

$$\lambda(t) = C_1 \cosh(\omega t \sin\alpha) + C_2 sh(\omega t \sin\alpha) + \frac{g\cos\alpha}{\omega^2 \sin^2\alpha} \quad (19.47)$$

The constants $C_1$ and $C_2$ can be determined by imposing the initial conditions. If these conditions are $\lambda = \lambda_0$ and $\dot{\lambda} = 0$, it follows that:

$$C_1 = \lambda_0 - \frac{g\cos\alpha}{\omega^2 \sin^2\alpha}; \; C_2 = 0 . \quad (19.48)$$

The motion of *A* is defined by the equation:

$$\lambda = \left(\lambda_0 - \frac{g\cos\alpha}{\omega^2 \sin^2\alpha}\right) ch(\omega \sin\alpha)t + \frac{g\cos\alpha}{\omega^2 \sin^2\alpha} . \quad (19.49)$$

An interesting case corresponds to the initial conditions:

$$\lambda_0 = \frac{g\cos\alpha}{\omega^2 \sin^2\alpha}; \quad \dot{\lambda} = 0 . \quad (19.50)$$

It follows $\lambda(t) = \frac{g\cos\alpha}{\omega^2 \sin^2\alpha}$ which indicates that the material point is at relative rest with respect to the bar.

Example 3. Write the Lagrange equations for a homogeneous disc of mass *M* and radius *R*, moving on an inclined surface (angle *α* with the horizontal direction). The contact is with sliding friction of coefficient *μ* and rolling friction of coefficient *s*. The rolling motion can be accompanied by sliding or it can be a pure rolling.

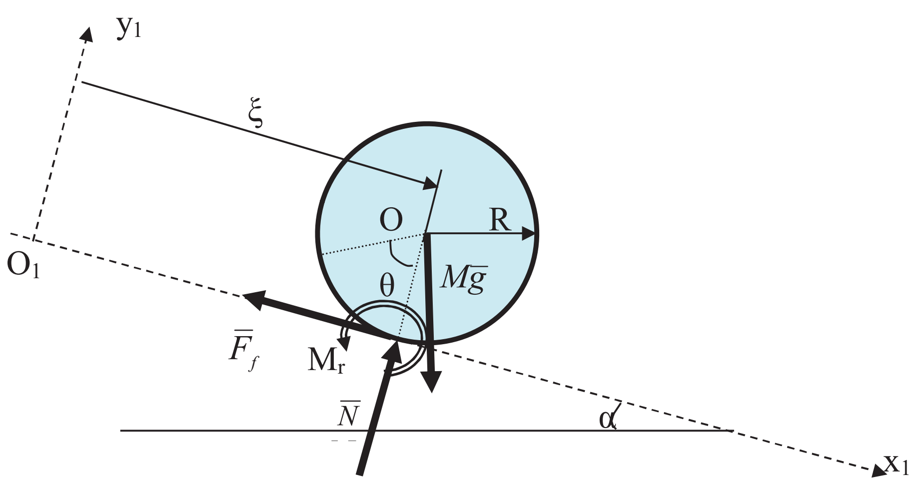



**Fig. 19.3 A disc moving on an inclined plane**

There are two degrees of freedom in the first case: ξ and θ and the system is holonomic. The kinetic energy is

$$T = \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}\frac{MR^2}{2}\dot{\theta}^2 \tag{19.51}$$

and the successive derivatives of the energy are:

$$\begin{aligned} &\frac{\partial T}{\partial \dot{\xi}} = M\dot{\xi};\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) = M\ddot{\xi}; \qquad \frac{\partial T}{\partial \xi} = 0; \\ &\frac{\partial T}{\partial \dot{\theta}} = \frac{MR^2}{2}\dot{\theta};\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{MR^2}{2}\ddot{\theta};\ \frac{\partial T}{\partial \theta} = 0. \end{aligned} \tag{19.52}$$

$$\frac{\partial T}{\partial \dot{\xi}} = M\dot{\xi};\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) = M\ddot{\xi};\ \frac{\partial T}{\partial \xi} = 0; \quad \frac{\partial T}{\partial \dot{\theta}} = \frac{MR^2}{2}\dot{\theta};\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{MR^2}{2}\ddot{\theta};\ \frac{\partial T}{\partial \theta} = 0;$$

The work is produced by: the weight moving down, the sliding friction force $F_f$ applied in the contact point which has an instantaneous velocity $\dot{\xi} - R\dot{\theta}$ and the rolling friction moment $M_r$ which is acting against the instantaneous rotation of angle θ:

$$\begin{aligned} \delta W &= mg\sin\alpha\delta\xi - F_f\left(\delta\xi - R\delta\theta\right) - M_r\delta\theta \\ &= \left(mg\sin\alpha - F_f\right)\delta\xi + \left(F_f R - M_r\right)\delta\theta \end{aligned} \tag{19.53}$$

from which

$$\begin{aligned} Q_\xi &= mg\sin\alpha - F_f \\ Q_\theta &= F_f R - M_r \end{aligned} \tag{19.54}$$

Since sliding is assumed to take place, then $F_f = \mu Mg\cos\alpha$ and in any case $M_r = sMg\cos\alpha$. The Lagrange equations are:

$$\begin{cases} M\ddot{\xi} = Mg\sin\alpha - F_f = Mg\left(\sin\alpha - \mu\cos\alpha\right) \\ \dfrac{MR^2}{2}\ddot{\theta} = F_r R - M_r = Mg\left(R\mu - s\right)\cos\alpha \end{cases} \tag{19.55}$$

For rolling without sliding, the contact point has instantaneous null velocity $\dot{\xi} - R\dot{\theta} = 0$ or $R\ddot{\theta} = \ddot{\xi}$.

For null initial conditions $\xi - R\theta = 0$, meaning that the system is in this case non-holonomic: $\delta\xi - R\delta\theta = 0$.

From (19.7) *s=1* and $a_{11}=1$; $a_{12}=-R$. The Lagrange equations are in this case

$$\begin{cases} M\ddot{\xi} = Mg\left(\sin\alpha - \mu\cos\alpha\right) + \lambda_1 \\ \dfrac{MR^2}{2}\ddot{\theta} = Mg\left(R\mu - s\right)\cos\alpha - \lambda_1 R \end{cases} \tag{19.56}$$

Eliminating $\lambda_1$ from these equations one gets a single Lagrange equation, since the system remains with only one degree of freedom:

$$\ddot{\xi} = \frac{2}{3} g \left( \sin\alpha - \frac{s}{R} \cos\alpha \right) \tag{19.57}$$

which corresponds to the equation (15.118) from chapter 15.

# 20. HAMILTON CANONICAL EQUATIONS

## 20.1. Integration of Lagrange equations. Hamilton Function

The Lagrange equations for a holonomic system are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0; \qquad k = 1,...,h \tag{20.1}$$

It can be assumed that

$$\frac{\partial L}{\partial q_k} = 0, \tag{20.2}$$

for a certain $q_k$. Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0. \tag{20.3}$$

Equation (20.2) is a differential equation of the first order. The Lagrange generalized coordinates $q_k$ for which the conditions (20.2) is accomplished, are called **cyclic coordinates**.

Therefore, if a generalized coordinate $q_k$ is cyclic, then its corresponding equations in Lagrange equation (20.1) can be replaced by a differential equation (20.3) of the first order. Since L = T + U and if U does not depend on $\dot{q}_1,...,\dot{q}_h$ it follows that:

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\dot{q}_k} = p_k; \quad k = 1,...,h \tag{20.4}$$

The physical quantity $p_k$ is called **generalized momentum**. It may be a linear momentum, an angular momentum or a more complicated quantity.

Example. Study the plane motion of a material point in polar coordinates r and θ, expressing the generalized momentums.
Since $v_\rho = \dot{r}$ and $v_n = r\dot{\theta}$ the expression of the kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_\rho^2 + v_n^2\right) = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] \tag{20.5}$$

The expression of the generalized momentum $p_r$ and $p_\theta$ are:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} = mv_\rho; \qquad p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} = mrv_n \tag{20.6}$$

Obviously, $p_r$ is a linear momentum and $p_\theta$ is an angular momentum.

Remark. The relation (20.3) can be regarded as a conservation property of the generalized momentum. Therefore, if some generalized coordinate $q_k$ is cyclic, then its corresponding generalized momentum $p_k$ is conservative:

$$p_k = C . \tag{20.7}$$

The relation (20.7) deduced in Analytic Mechanics corresponds to two conservation theorems from Newtonian Mechanics: conservation of linear momentum and conservation of angular momentum.
The conditions for conservation theorem of total energy are now investigated. Multiplying each equation (20.1) by $\dot{q}_k$ and adding afterwards the obtained equations it can be obtained:

$$\sum_{k=1}^{h} \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial q_k} = 0 . \tag{20.8}$$

The terms in the first sum can be developed as:

$$\sum_{k=1}^{h} \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \sum_{k=1}^{h} \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{k=1}^{h} \ddot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \tag{20.9}$$

so that (20.8) can be written:

$$\sum_{k=1}^{h} \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{k=1}^{h} \ddot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial q_k} = 0. \tag{20.10}$$

The total derivative of the Lagrange kinetic potential $L$ can be written as:

$$\frac{dL}{dt} = \sum_{k=1}^{h} \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_{k=1}^{h} \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t}. \tag{20.11}$$

Using this development, the expression (20.10) implies:

$$\sum_{k=1}^{h} \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) - \frac{dL}{dt} + \frac{\partial L}{\partial t} = \frac{d}{dt}\left[\sum_{k=1}^{h}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right)\right] - \frac{dL}{dt} + \frac{\partial L}{\partial t} = 0. \tag{20.12}$$

Replacing (20.4) it follows

$$\frac{d}{dt}\left[\sum_{k=1}^{h}(p_k \dot{q}_k) - L\right] + \frac{\partial L}{\partial t} = 0. \tag{20.13}$$

If $\frac{\partial L}{\partial t} = 0$, a first integral is obtained:

$$H = \sum_{k=1}^{h} p_k \dot{q}_k - L = const. \tag{20.14}$$

The function $H$ thus defined is known as **Hamilton's function**.

It is easy to prove that *H=E,* with E being the total energy of a mechanical system, if the system of material points is scleronomic and holonomic $\left(\frac{\partial \mathrm{L}}{\partial \mathrm{t}}=0\right)$.

Indeed, in this case the kinetic energy T is a quadratic form in variables $\dot{q}_1,...,\dot{q}_h$, which can be written as:

$$T=\sum_{i,k=1}^{h} a_{ik}\dot{q}_i\dot{q}_k\ . \tag{20.15}$$

A well-known property of homogeneous functions $f(x_1,....,x_h)$ of degree *n* is the following:

$$x_1\frac{\partial f}{\partial x_1}+...+x_h\frac{\partial f}{\partial x_h}=nf(x_1,...,x_h) \tag{20.16}$$

which can be applied to the function *T*, for which n = 2. It follows that:

$$\frac{\partial T}{\partial \dot{q}_1}\dot{q}_1+...+\frac{\partial T}{\partial \dot{q}_h}\dot{q}_h=2T\ . \tag{20.17}$$

Since $\frac{\partial T}{\dot{q}_k}=p_k$, this implies:

$$\sum_{k=1}^{h} p_k\dot{q}_k=2T\ . \tag{20.18}$$

The Lagrange kinetic potential $L=T+U=T-V$, where *V* is the potential energy, the expression (20.14) of H becomes:

$$H=\sum_{k=1}^{h} p_k\dot{q}_k-L=2T-T+V=T+V=E=const. \tag{20.19}$$

which represents the theorem of total energy balance (conservation). Note that *H* = *T* + *V* = *E* only for scleronomic holonomic systems.

### *20.2. Hamilton Canonical Equations*

Returning to the relation (20.4):

$$\frac{\partial T}{\dot{q}_k}=p_k;\quad k=1,...,h, \tag{20.20}$$

these expressions can be considered as a linear system of equations in $\dot{q}_1,...,\dot{q}_h$. Supposing the solution of this system as having the form:

$$\dot{q}_k=\dot{q}_k(p_1,...,p_h,q_1,...,q_h,t);\ \ k=1,...,h, \tag{20.21}$$

and replacing $\dot{q}_k(k=1,...,h)$ in (20.14):

$$H=\sum_{k=1}^{h} p_k \dot{q}_k - L(p_1,...,p_h,q_1,...,q_h,t), \qquad (20.22)$$

it follows that the expression of H is a function of $p_1,...p_h,q_1,...,q_h,t$ only:

$$H(p_1,...,p_h,q_1,...,q_h,t). \qquad (20.23)$$

The partial derivatives of H with respect to $p_k$ and $q_k$, taking into account (20.4), (20.1) and (20.20), can be obtained as follows:

$$\frac{\partial H}{\partial p_k}=\dot{q}_k+\sum_{j=1}^{h} p_j \frac{\partial \dot{q}_j}{\partial p_k}-\sum_{j=1}^{h}\frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_k}=\dot{q}_k+\sum_{j=1}^{h} p_j \frac{\partial \dot{q}_j}{\partial p_k}-\sum_{j=1}^{h} p_j \frac{\partial \dot{q}_j}{\partial p_k}=\dot{q}_k.$$

$$\frac{\partial H}{\partial q_k}=\sum_{j=1}^{h} p_j \frac{\partial \dot{q}_j}{\partial q_k}-\frac{\partial L}{\partial q_k}-\sum_{j=1}^{h}\frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_k}=\sum_{j=1}^{h} p_j \frac{\partial \dot{q}_j}{\partial q_k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\sum_{j=1}^{h} p_j \frac{\partial \dot{q}_j}{\partial q_k}= \qquad (20.24)$$

$$=-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)=-\frac{dp_k}{dt}=-\dot{p}_k.$$

Consequently:

$$\frac{\partial H}{\partial p_k}=\dot{q}_k; \qquad \frac{\partial H}{\partial q_k}=-\dot{p}_k. \qquad (20.25)$$

These equations are called the **Hamilton canonical equations**.

An interesting remark can be made concerning the derivative of H with respect to the time:

$$\frac{dH}{dt}=\sum_{j=1}^{h}\frac{\partial H}{\partial p_k}\dot{p}_k+\sum_{j=1}^{h}\frac{\partial H}{\partial q_k}\dot{q}_k+\frac{\partial H}{\partial t}=\sum_{j=1}^{h}\frac{\partial H}{\partial p_k}\left(-\frac{\partial H}{\partial q_k}\right)+\sum_{j=1}^{h}\frac{\partial H}{\partial q_k}\frac{\partial H}{\partial p_k}+\frac{\partial H}{\partial t}=\frac{\partial H}{\partial t} \qquad (20.26)$$

indicating that the derivative of the Hamilton function with respect to time is equal to its partial derivative with respect to time.
An immediate consequence is that in scleronomic – holonomic and autonomous systems for which $\frac{\partial H}{\partial t}=0$, the Hamilton function is a constant. Since in such systems: $H = T + V = E$, it is again found the principle of the total energy E balance (conservation) in scleronomic – holonomic systems.

Example.
Write the canonical equations of Hamilton for the motion of a material of mass *m* point acted by a Newtonian force of universal attraction from a material point of mass M situated at distance r.
The kinetic energy is:

$$T=\frac{1}{2}mv^2=\frac{1}{2}m\left(v_\rho^2+v_n^2\right)=\frac{1}{2}m\left[\dot{r}^2+r^2\dot{\theta}^2\right]. \qquad (20.27)$$

The potential energy is:

$$V = -f\frac{mM}{r}, \tag{20.28}$$

in which $f$=6.67428e-11 $m^3kg^{-1}s^{-2}$ is the universal attraction constant. The Hamilton function is:

$$H = T + V = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - f\frac{mM}{r} \tag{20.29}$$

The expressions of the generalized momentums are:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}; \qquad p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{20.30}$$

Solving this system of equations, it follows that:

$$\dot{r} = \frac{p_r}{m}; \qquad \dot{\theta} = \frac{p_\theta}{mr^2} \tag{20.31}$$

Substituting these expressions in $H$, it can be obtained:

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{fmM}{r} \tag{20.32}$$

The Hamilton canonical equations, in general form:

$$\dot{p}_r = \frac{\partial H}{\partial r}; \;\; \dot{p}_\theta = -\frac{\partial H}{\partial \theta}; \;\; \dot{r} = \frac{\partial H}{\partial p_r}; \;\; \dot{\theta} = -\frac{\partial H}{\partial p_\theta} \tag{20.33}$$

become in this case:

$$\dot{p}_r = -\frac{p_\theta^2}{mr^3} + \frac{fmM}{r^2}; \quad \dot{p}_\theta = 0; \quad \dot{r} = \frac{p_r}{m}; \quad \dot{\theta} = \frac{p_\theta}{mr^2}. \tag{20.34}$$

### *20.3. Integration of Hamilton canonical equations. Poisson brackets*

In order to integrate Hamilton canonical equations, it can be sought as first (prime) integrals which are constant expressions of the form:

$$F(p_1,...,p_h,q_1,...,q_h,t) = C \text{ or } \frac{dF}{dt} = 0. \tag{20.35}$$

It is necessary to calculate the derivative with respect to time $t$ of such a function $F(p_1,...,p_h,q_1,...,q_h,t)$. It follows that:

$$\frac{dF}{dt} = \sum_{j=1}^{h}\frac{\partial F}{\partial p_k}\dot{p}_k + \sum_{j=1}^{h}\frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t} = \sum_{j=1}^{h}\left(\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right) + \frac{\partial F}{\partial t} \tag{20.36}$$

The following notation is introduced:

$$\sum_{j=1}^{h}\left(\frac{\partial\Phi}{\partial q_k}\frac{\partial\Psi}{\partial p_k}-\frac{\partial\Phi}{\partial p_k}\frac{\partial\Psi}{\partial q_k}\right)=(\Phi,\Psi), \tag{20.37}$$

where $\Phi$ and $\Psi$ are functions of $p_1,....,p_h,q_1,...,q_h,t$. The expression $(\Phi,\Psi)$ is called the **Poisson bracket** of functions $\Phi$ and $\Psi$. Using this notation, (20.36) becomes:

$$\frac{dF}{dt}=(F,H)+\frac{\partial F}{\partial t}. \tag{20.38}$$

The Poisson bracket has some remarkable properties. It is easy to verify that

$$\begin{gathered}(\Phi,C)=0;\quad (\Phi,\Psi)=-(\Psi,\Phi);\quad (C\Phi,\Psi)=C(\Phi,\Psi);\\ \frac{\partial(\Phi,\Psi)}{\partial t}=\left(\frac{\partial\Phi}{\partial t},\Psi\right)+\left(\Phi,\frac{\partial\Psi}{\partial t}\right);\quad (p_k,H)=-\frac{\partial H}{\partial q_k};\quad (q_k,H)=\frac{\partial H}{\partial p_k}\end{gathered} \tag{20.39}$$

Using Poisson bracket, Hamilton canonical equations, can be expressed as:

$$\dot{p}_k=(p_k,H);\quad \dot{q}_k=(q_k,H). \tag{20.40}$$

Another property of the Poisson brackets is the **Poisson-Jacobi identity**:

$$((P,\ Q),\ R)\ +\ ((Q,\ R),\ P)\ +\ ((R,\ P),\ Q)\ =0. \tag{20.41}$$

### 20.3.1. The Poisson theorem

If $\Phi = C_1$ and $\Psi = C_2$ are two first integrals of Hamilton canonical equations, then $(\Phi,\Psi)$ is also a first integral.

Indeed, if $\Phi = C_1$ and $\Psi = C_2$ are first integrals, then:

$$(\Phi,\ \mathrm{H})+\frac{\partial\Phi}{\partial t}=0;\quad (\Psi,\ \mathrm{H})+\frac{\partial\Psi}{\partial t}=0 \tag{20.42}$$

By using the Poisson-Jacobi identity (20.41), it follows:

$$((\Phi,\ \Psi),\ \mathrm{H})\ +((\Psi,\mathrm{H}),\Phi)+\ ((\mathrm{H},\Phi),\Psi)\ =\ 0 \tag{20.43}$$

The expressions (20.42) can be written:

$$(\Phi,\ \mathrm{H})=-\frac{\partial\Phi}{\partial t};\quad (\Psi,\ \mathrm{H})=-\frac{\partial\Psi}{\partial t} \tag{20.44}$$

and (20.43) can be expressed successively:

$$\begin{gathered}((\Phi,\ \Psi),\ \mathrm{H})\ +\left(-\frac{\partial\Psi}{\partial t},\Phi\right)+\ \left(\frac{\partial\Phi}{\partial t},\Psi\right)\ =\ 0\\ ((\Phi,\ \Psi),\ \mathrm{H})+\frac{\partial(\Phi,\Psi)}{\partial t}=0\end{gathered} \tag{20.45}$$

Therefore $(\Phi, \Psi)$ is an integral of the Hamilton canonical equations.

### *20.4. Variational Principles. Hamilton Principle*

While the theory of ordinary functions maxima and minima is concerned with unknown values of independent variables $x$ or $x_i$ corresponding to maxima and minima of given functions, it is the objective of the calculus of variations to find unknown functions $y(x)$ or $y_i(x)$; $i=1,\dots h,$ which will maximize or minimize definite integrals like:

$$I = \int_{x_1}^{x_2} F\left[y(x), y'(x), x\right] dx , \tag{20.46}$$

or

$$I = \int_{x_1}^{x_2} F\left[y_1(x), \dots, y_h(x), y_1'(x), \dots, y_h'(x); x\right] dx . \tag{20.47}$$

A necessary condition for the existence of either a maximum or a minimum of the definite integral (20.46) is that the function F verifies the differential equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 , \tag{20.48}$$

and for the integral (20.47), the equivalent conditions are:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right) - \frac{\partial F}{\partial y_k} = 0; \qquad k = 1, \dots, h \tag{20.49}$$

These differential equations are called the **Euler differential equations**. Comparing the Euler differential equations (20.49)with the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0; \quad k = 1, \dots, h \tag{20.50}$$

it follows that the Lagrange equations can be considered as Euler differential equations for the integral of the Lagrange kinetic potential :

$$I = \int_{t_1}^{t_2} L\left(q_1, \dots, q_h, \dot{q}_1, \dots, \dot{q}_h, t\right) dt \tag{20.51}$$

The quantity $I$ is a product of energy by time and is called **mechanical action** or simply **action**.

From the above statements it follows that the **action** represents a maximum or minimum or in general a stationary value, which can be expresses as:

$$\delta I = 0 \quad \text{or} \quad \delta\left(\int_{S_1}^{S_2} L dt\right) = 0 . \tag{20.52}$$

The operator $\delta$ corresponds to variations in the sense of virtual displacements and velocities in the Lagrange kinetic potential, evaluated from the initial state $S_1$ at time $t_1$ and the final state $S_2$ at time $t_2$. This formula can be expressed as:
"From all the possible motions of a mechanical system acted by non-dissipative forces (but the force function can depend on time), it will take place that particular motion which minimizes the mechanical action."
This statement is known as the **principle of the minimum action or Hamilton principle**.

Example.
Deduce that it is possible to deduce the Hamilton canonical equations from the Hamilton principle.
Indeed, from the expression of $H = \sum_{k=1}^{h} p_k \dot{q}_k - L$ it follows that:

$$L = \sum_{k=1}^{h} p_k \dot{q}_k - H \,. \tag{20.53}$$

The integral (20.51) can be written:

$$I = \int_{t_1}^{t_2} \left[ \sum_{k=1}^{h} p_k \dot{q}_k - H\left(p_1,\ldots,p_h,q_1,\ldots,q_h,t\right) \right] dt \,. \tag{20.54}$$

Considering as functions $p_1,\ldots,p_h$ and $q_1,\ldots,q_h$ and denoting by $F = \sum_{k=1}^{h} p_k \dot{q}_k - H$, then *Euler equations* are:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_k}\right) - \frac{\partial F}{\partial q_k} = 0; \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{p}_k}\right) - \frac{\partial F}{\partial p_k} = 0; \quad k = 1,\ldots,h \tag{20.55}$$

or

$$\frac{d}{dt}\left(p_k\right) - \left(-\frac{\partial H}{\partial q_k}\right) = 0; \quad \frac{d}{dt}(0) - \dot{q}_k - \left(-\frac{\partial H}{\partial p_k}\right) = 0; \tag{20.56}$$

from which if follows:

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}; \qquad \dot{q}_k = \frac{\partial H}{\partial p_k} \,. \tag{20.57}$$

and these are the Hamilton canonical equations.

### *20.5. Canonical Transformations*

There is no unique system of generalized coordinates for a given system of material points. Practically the problem is to choose such a system of generalized coordinates, that the Hamilton canonical equations will be as simple as possible. In order to solve this problem it is necessary to study the general problem of the generalized coordinates and generalized momentums:

$$\begin{aligned} q_k &= q_k\left(Q_1,\ldots Q_h,\ P_1,\ldots,P_h,t\right); \quad k=1,\ldots,h \\ p_k &= \mathrm{p}_k\left(Q_1,\ldots Q_h,\ P_1,\ldots,P_h,t\right); \quad k=1,\ldots,h \end{aligned} \tag{20.58}$$

Replacing these expressions of $q_k$ and $p_k$ in $H\ (p_1,\ldots,p_h,q_1,\ldots,q_h,t)$ a new expression $K(P_1,\ldots,P_h,\ Q_1,\ldots Q_h,t)$ of the Hamilton function can be obtained. It is important that the Hamilton canonical equations maintain the same form:

$$\dot{P}_k = -\frac{\partial K}{\partial Q_k}; \quad \dot{Q}_k = \frac{\partial K}{\partial P_k}; \quad k=1,\ldots,h. \tag{20.59}$$

Not all transformations (20.58) maintain the form (20.59) for the Hamilton canonical equations. The transformations which maintain this form are called **canonical transformations.**

It is possible to prove that a transformation (20.58) is canonical if and only if:

$$\sum_{k=1}^{h} P_k dQ_k = \sum_{k=1}^{h} p_k dq_k + d\Psi, \tag{20.60}$$

where $\Psi$ is a function of $P_1,\ldots,P_h,\ Q_1,\ldots Q_h$.

Indeed, Hamilton canonical equations can be deduced from the Hamilton principle:

$$\delta I = \delta\int_{t_1}^{t_2}\left[\sum_{k=1}^{h} p_k\dot{q}_k - H[p_1,\ldots,p_h,q,\ldots,q_h,t_1]\right]dt. \tag{20.61}$$

If $p_k$ and $q_k$ are replaced by their expressions (20.58), and if the condition (20.60) is accomplished, then the integral $I$ becomes:

$$\int_{t_1}^{t_2}\left[\sum_{k=1}^{h} p_k\dot{Q}_k - K\left(P_1,\ldots,P_h,Q_1,\ldots,Q_h,t\right) + \frac{d\psi}{dt}\right]dt, \tag{20.62}$$

but

$$\int_{t_1}^{t_2} d\psi = \psi_2 - \psi_1 = const., \tag{20.63}$$

so that the minimum of the integral in (20.62) takes place simultaneously with the minimum of the integral:

$$I = \int_{t_1}^{t_2}\left[\sum_{k=1}^{h} P_k\dot{Q}_k - K\left(P_1,\ldots,P_h,Q_1,\ldots,Q_h,t\right)\right]dt. \tag{20.64}$$

The corresponding Euler differential equations are:

$$\dot{P}_k = -\frac{\partial K}{\partial Q_k}; \quad \dot{Q}_k = \frac{\partial K}{\partial P_k}; \quad k=1,\ldots,h, \tag{20.65}$$

which are representing the Hamilton canonical equations.

Example.
Prove that the following transformation is canonical:

$$P = \sqrt{C+2p}\cdot\cos q; \quad Q = \sqrt{C+2p}\cdot\sin q\,, \tag{20.66}$$

in which C is a constant.

It has to be proven that $PdQ - pdq = d\Psi$, which means the left side of the equality is an exact differential:

$$\begin{aligned} PdQ - pdq &= \sqrt{C+2P}\cdot\cos Q\cdot\left[\frac{dp}{\sqrt{C+2p}}\sin q + \sqrt{C+2p}\cos q dq\right] - pdq \\ &= \sin q\cos q dp + \left[\left(C+2p\right)\cos^2 q - p\right]dq = \frac{1}{2}d\left[p\sin 2q + C\left(q + \frac{\sin 2q}{2}\right)\right] \end{aligned} \tag{20.67}$$

It follows that

$$\Psi = \frac{1}{2}\left[p\sin 2q + C\left(q + \frac{\sin 2q}{2}\right)\right], \tag{20.68}$$

and therefore the transformation is canonical.

### 20.5.1. Lagrange Brackets

The condition (20.60) for a canonical transformation is:

$$\sum_{k=1}^{h} P_k dQ_k = \sum_{j=1}^{h} p_j dq_j + d\Psi\,, \tag{20.69}$$

in which the new variables $P_k$, $Q_k$ will be expressed as functions of the old variables $p_k$, $q_k$ . Thus

$$\begin{aligned} dQ_k &= \sum_{j=1}^{h}\frac{\partial Q_k}{\partial p_j}dp_j + \sum_{j=1}^{h}\frac{\partial Q_k}{\partial q_j}dq_j; \quad k = 1,...,h \\ d\Psi &= \sum_{j=1}^{h}\frac{\partial\psi}{\partial p_j}dp_j + \sum_{j=1}^{h}\frac{\partial\psi}{\partial q_j}dq_j \end{aligned} \tag{20.70}$$

Substituting these expressions of $dQ_k$ and $d\Psi$ in (20.69), it follows:

$$\sum_{k=1}^{h} P_k dQ_k = \sum_{k=1}^{h} P_k\left(\sum_{j=1}^{h}\frac{\partial Q_k}{\partial p_j}dp_j + \sum_{j=1}^{h}\frac{\partial Q_k}{\partial q_j}dq_j\right) = \sum_{k=1}^{h} p_k dq_k + \sum_{j=1}^{h}\frac{\partial\psi}{\partial p_j}dp_j + \sum_{j=1}^{h}\frac{\partial\psi}{\partial q_j}dq_j \tag{20.71}$$

and grouping adequately the terms, it follows:

$$\sum_{j=1}^{h}\left(\sum_{k=1}^{h} P_k\frac{\partial Q_k}{\partial p_j} - \frac{\partial\psi}{\partial p_j}\right)dp_j - \sum_{j=1}^{h}\left(p_j + \frac{\partial\psi}{\partial q_j} - \sum_{k=1}^{h} P_k\frac{\partial Q_k}{\partial q_j}\right)dq_j = 0\,. \tag{20.72}$$

Since $dp_j$ and $dq_j$ are independent, the relation (20.72) is equivalent to the following *h* systems of equations:

$$\sum_{k=1}^{h} P_k \frac{\partial Q_k}{\partial p_j} - \frac{\partial \psi}{\partial p_j}; \qquad j = 1,...,h$$
$$\sum_{k=1}^{h} P_k \frac{\partial Q_k}{\partial q_j} - \frac{\partial \psi}{\partial q_j} = p_j; \quad j = 1,...,h \tag{20.73}$$

The function $\Psi$ can be eliminated from the relations (20.73). There are several possibilities.

a) From the first set of equations, it will be written the equation for $j=i$ ($i \in [1,...,n]$) and then derivate with respect to $p_l$. From the same set it will be written the equation for $j=l$ with $l \in [1,...,n]$ and derivate this relation with respect to $p_i$. Subtracting the obtained equations, one gets:

$$\sum_{k=1}^{h} \left( \frac{\partial P_k}{\partial p_l} \frac{\partial Q_k}{\partial p_i} + P_k \frac{\partial^2 Q_k}{\partial p_l \partial p_i} \right) - \sum_{k=1}^{h} \left( \frac{\partial P_k}{\partial p_i} \frac{\partial Q_k}{\partial p_l} + P_k \frac{\partial^2 Q_k}{\partial p_i \partial p_l} \right) = 0, \tag{20.74}$$

or after cancelling identical terms:

$$[p_l, p_i] = \sum_{k=1}^{h} \left[ \frac{\partial P_k}{\partial p_l} \frac{\partial Q_k}{\partial p_i} - \frac{\partial P_k}{\partial p_i} \frac{\partial Q_k}{\partial p_l} \right] = 0. \tag{20.75}$$

b) From the second set of equations (20.73), written for $j=i$ , then computing its derivative with respect to $q_l$ and repeating the procedure for $j=l$ and computing the derivative with respect to $q_i$. Subtracting the equations thus obtained, one gets:

$$\sum_{k=1}^{h} \left( \frac{\partial P_k}{\partial q_l} \frac{\partial Q_k}{\partial q_i} + P_k \frac{\partial^2 Q_k}{\partial q_l \partial q_i} \right) - \sum_{k=1}^{h} \left( \frac{\partial P_k}{\partial q_i} \frac{\partial Q_k}{\partial q_l} + P_k \frac{\partial^2 Q_k}{\partial q_i \partial q_l} \right) = 0, \tag{20.76}$$

or after cancelling identical terms:

$$[q_l, q_i] = \sum_{k=1}^{h} \left[ \frac{\partial P_k}{\partial q_l} \frac{\partial Q_k}{\partial q_i} - \frac{\partial P_k}{\partial q_i} \frac{\partial Q_k}{\partial q_l} \right] = 0. \tag{20.77}$$

c) From the first set of equations (20.73), written for $j=i$ is computed its derivative with respect to $q_l$ and from the second set of equations written for $j=l$ is computed the derivative with respect to $p_i$. Subtracting the equations thus obtained, it can be obtained:

$$\sum_{k=1}^{h} \left( \frac{\partial P_k}{\partial p_i} \frac{\partial Q_k}{\partial q_l} + P_k \frac{\partial^2 Q_k}{\partial p_i \partial q_l} \right) - \sum_{k=1}^{h} \left( \frac{\partial P_k}{\partial q_l} \frac{\partial Q_k}{\partial p_i} + P_k \frac{\partial^2 Q_k}{\partial q_l \partial p_i} \right) = \delta_{il}, \tag{20.78}$$

or after cancelling identical terms:

$$[p_i, q_l] = \sum_{k=1}^{h} \left( \frac{\partial P_k}{\partial p_i} \frac{\partial Q_k}{\partial q_l} - \frac{\partial P_k}{\partial q_l} \frac{\partial Q_k}{\partial p_i} \right) = \delta_{il}, \tag{20.79}$$

in which $\delta_{il}$ is the *Kronecker symbol* ($\delta_{il} = 0$ if $l \neq i$ and $\delta_{il} = 1$ if $i = l$).
These expressions (20.75), (20.77) and (20.79) are called **Lagrange brackets**.

Example. Prove that the following transformation is canonical:

$$P = \sqrt{C+2p}\cdot\cos q; \quad Q = \sqrt{C+2p}\cdot\sin q\,, \tag{20.80}$$

in which C is a constant.

In this case there is an unique Lagrange bracket $[q;p]$ which must be equal to 1. It follows that:

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \sqrt{C+2p}\cos q\frac{1}{\sqrt{C+2p}}\cos q - \sqrt{C+2p}\sin q\frac{1}{\sqrt{C+2p}}(-\sin q) \tag{20.81}$$
$$= \cos^2 q + \sin^2 q = 1$$

## *20.6. Phase Space*

The generalized coordinates $q_1,...,q_h$ and the generalized momentums $p_1,...,p_h$ can be considered as the coordinates of a point in a space with $2h$ dimensions. This space was introduced by Gibbs. It is called **phase space**. The motion of a system of material points can be represented by a path in the phase space. A point in the phase space does not represent the position of the material point system only, but also the generalized momentums of the system. Therefore a point in the phase space represents **the state** of the system of material points.

Example. Determine the locus in the phase space for the motion of a material point of mass $m$ acted by an elastic force proportional with the position by an elastic constant $k$.

The kinetic and potential energy are respectively:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2; \quad V = \frac{1}{2}kx^2 \tag{20.82}$$

The *Hamilton function* is:

$$H = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{20.83}$$

The *generalized momentum* is:

$$p = \frac{\partial T}{\partial \dot{x}} = m\dot{x}\,. \tag{20.84}$$

By denoting $x = q$, the Hamilton function can be written:

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2\,. \tag{20.85}$$

Since the system is autonomous, $H = C$ with $C$ a constant which can be obtained from the initial conditions. Thus (20.85) can be cast into the following form:

$$\frac{p^2}{2m}+\frac{1}{2}kq^2=C \;\Rightarrow\; \frac{q^2}{\left(\sqrt{\frac{2C}{k}}\right)^2}+\frac{p^2}{\left(\sqrt{2mC}\right)^2}=1. \qquad (20.86)$$

The path in the phase plane is an "ellipse". The position "semi-axis" is $\sqrt{\frac{2C}{k}}$ and the momentum semi-axis is $\sqrt{2mC}$. It is obvious the semi-axis have different dimensions, but the locus or the "path" in this phase space gives a rapid overall impression about the motion taking place. An ellipse is a perpetual path which is typical for non-dissipative systems. A path converging towards a point represents a dissipating system and so on.

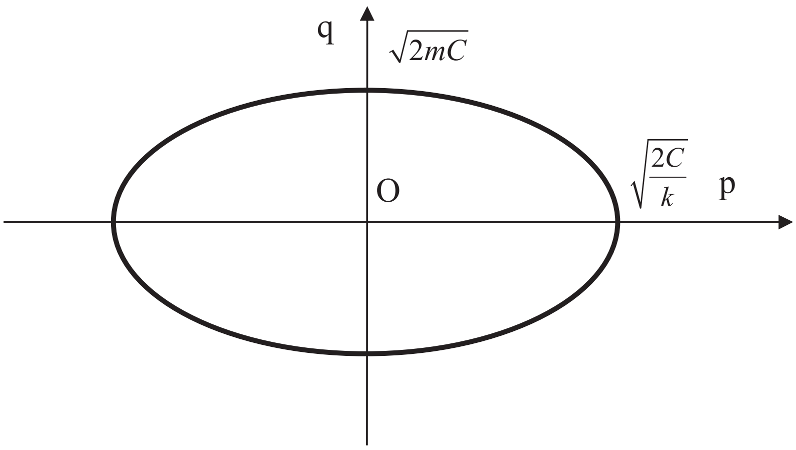



**Fig. 20.1 The path in the phase plane for a material point acted by an elastic force**

### 20.6.1. Liouville Theorem

Supposing that a system of material points is characterized by the generalized coordinates $q_k$ and generalized momentums $p_k$ at a time t, that becomes $Q_k$ and $P_k$ at a time $t+dt$ it can be written:

$$\begin{aligned} Q_k &= q_k+\dot{q}_k dt = q_k+\frac{\partial H}{\partial p_k}dt; \qquad k=1,\ldots,h \\ P_k &= p_k+\dot{p}_k dt = p_k-\frac{\partial H}{\partial q_k}dt; \qquad k=1,\ldots,h \end{aligned} \qquad (20.87)$$

It can be proven that (20.87) defines a canonical transformation, such that:

$$\sum_{k=1}^{h}P_k\dot{Q}_k-\sum_{k=1}^{h}p_k\dot{q}_k=d\Psi. \qquad (20.88)$$

Indeed, using (20.60), these transformations can be written as:

$$\begin{aligned}
\sum_{k=1}^{h}\left[P_k\dot{Q}_k - p_k\dot{q}_k\right] &= \sum_{k=1}^{h}\left[\left(p_k - \frac{\partial H}{\partial q_k}dt\right)\left[\dot{q}_k + \frac{d}{dt}\left(\frac{\partial H}{\partial p_k}\right)dt\right] - p_k\dot{q}_k\right] \\
&= \sum_{k=1}^{h}\left[p_k\frac{d}{dt}\left(\frac{\partial H}{\partial p_k}\right)dt - \dot{q}_k\frac{\partial H}{\partial q_k}dt\right] = \sum_{k=1}^{h}\left[\frac{d}{dt}\left(p_k\frac{\partial H}{\partial p_k}\right) - \frac{\partial H}{\partial p_k}\dot{p}_k - \dot{q}_k\frac{\partial H}{\partial q_k}\right]dt \\
&= \sum_{k=1}^{h}\left[\frac{d}{dt}\left(p_k\frac{\partial H}{\partial p_k}\right)\right]dt - \sum_{k=1}^{h}\left[\frac{\partial H}{\partial p_k}\dot{p}_k + \dot{q}_k\frac{\partial H}{\partial q_k}\right]dt \\
&= d\left(\sum_{k=1}^{h}p_k\frac{\partial H}{\partial p_k}\right) - \sum_{k=1}^{h}\left[-\frac{\partial H}{\partial q_k}\frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial p_k}\frac{\partial H}{\partial q_k}\right]dt = d\left(\sum_{k=1}^{h}p_k\frac{\partial H}{\partial p_k}\right) = d\psi
\end{aligned} \tag{20.89}$$

in which

$$\psi = \sum_{i=1}^{h}p_k\frac{\partial H}{\partial p_k}. \tag{20.90}$$

It follows that the motion of a system of material points can be represented in the phase space as a series of infinitesimal canonical transformations.
Consider a domain ($D$) in the phase space and all the paths that begin in points of this domain at time $t$. Let be ($D'$) be the domain occupied by the same points at time $t + dt$. It must be proven that the volume of $D$ is equal to the volume of D':

$$\int_D\ldots\int dp_1\ldots dp_h dq_1\ldots dq_h = \int_{D'}\ldots\int dP_1\ldots dP_h dQ_1\ldots dQ_h \tag{20.91}$$

Indeed

$$dP_1\ldots dP_h dQ_1\ldots dQ_h = |J|\, dp_1\ldots dp_h dq_1\ldots dq_h \tag{20.92}$$

in which

$$|J| = \begin{vmatrix}
\frac{\partial P_1}{\partial p_1} & \ldots & \frac{\partial P_1}{\partial p_h} & \frac{\partial P_1}{\partial q_1} & \ldots & \frac{\partial P_1}{\partial q_h} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{\partial P_h}{\partial p_1} & \ldots & \frac{\partial P_h}{\partial p_h} & \frac{\partial P_h}{\partial q_1} & \ldots & \frac{\partial P_h}{\partial q_h} \\
\frac{\partial Q_1}{\partial p_1} & & \frac{\partial Q_1}{\partial p_h} & \frac{\partial Q_1}{\partial q_1} & & \frac{\partial Q_1}{\partial q_h} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{\partial Q_h}{\partial p_1} & \ldots & \frac{\partial Q_h}{\partial p_h} & \frac{\partial Q_h}{\partial q_1} & \ldots & \frac{\partial Q_h}{\partial q_h}
\end{vmatrix} = \begin{vmatrix}
1+\frac{\partial \dot{p}_1}{\partial p_1}dt & \ldots & \frac{\partial \dot{p}_1}{\partial p_h}dt & \frac{\partial \dot{p}_1}{\partial q_1}dt & \ldots & \frac{\partial \dot{p}_1}{\partial q_h}dt \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{\partial \dot{p}_h}{\partial p_1}dt & \ldots & 1+\frac{\partial \dot{p}_h}{\partial p_h}dt & \frac{\partial \dot{p}_h}{\partial q_1}dt & \ldots & \frac{\partial \dot{p}_h}{\partial q_h}dt \\
\frac{\partial \dot{q}_1}{\partial p_1}dt & & \frac{\partial \dot{q}_1}{\partial p_h}dt & 1+\frac{\partial \dot{q}_1}{\partial q_1}dt & & \frac{\partial \dot{q}_1}{\partial q_h}dt \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{\partial \dot{q}_h}{\partial p_1}dt & \ldots & \frac{\partial \dot{q}_h}{\partial p_h}dt & \frac{\partial \dot{q}_h}{\partial q_1}dt & \ldots & 1+\frac{\partial q_1}{\partial q_h}dt
\end{vmatrix} \tag{20.93}$$

$$= 1 + \sum_{k=1}^{h}\left(\frac{\partial \dot{p}_k}{\partial p_k} + \frac{\partial \dot{q}_k}{\partial q_k}\right)dt + O(dt^2)$$

Considering the fact that

$$\sum_{k=1}^{h}\left(\frac{\partial \dot{p}_k}{\partial p_k}+\frac{\partial \dot{q}_k}{\partial q_k}\right)dt=\sum_{k=1}^{h}\left(\frac{\partial}{\partial p_k}\left(-\frac{\partial H}{\partial q_k}\right)+\frac{\partial}{\partial q_k}\left(\frac{\partial H}{\partial p_k}\right)\right)dt=0, \tag{20.94}$$

it follows that $|J|=1+O(dt^2)$ and therefore

$$dP_1...dP_h dQ_1...dQ_h = dp_1...dp_h dq_1...dq_h, \tag{20.95}$$

proving thus the equality (20.91).
The volume of a domain $D$ of the phase space is an integral invariant to the Hamilton canonical equations. This statement is known as the **theorem of Liouville**.

## *20.7. The Hamilton-Jacobi partial differential equation*

The following partial differential equation of the first order is considered:

$$\frac{\partial S}{\partial t}+H\left(q_1,...,q_h,\frac{\partial S}{\partial q_1},...,\frac{\partial S}{\partial q_h},t\right)=0, \tag{20.96}$$

obtained by replacing in the Hamilton function $H(q_1,...,q_h,p_1,...,p_h)$ the generalized momentums $p_k$ by the partial derivatives $\frac{\partial S}{\partial q_k}$ of a function $S = S(q_1,...,q_h)$.
This equation is called the **Hamilton - Jacobi partial differential equation**.

### 20.7.1. Jacobi theorem

If $S = S(q_1,...,q_h, a_1,...,a_h,t) + a_{h+1}$ is the complete integral of (20.96), where $a_1,...,a_h$, are essential constants and $a_{h+1}$ is an additive constant such that:

$$\det\left[\frac{\partial^2 S}{\partial q_k \partial a_i}\right]\neq 0, \tag{20.97}$$

then the expressions of $q_k = q_k(t)$, $p_k = p_k(t)$, $k = 1,...,h$ can be obtained from the following relations:

$$\begin{aligned}&\frac{\partial S}{\partial a_1}=b_1, ..., \frac{\partial S}{\partial a_h}=b_h\\&\frac{\partial S}{\partial q_1}=p_1, ..., \frac{\partial S}{\partial q_h}=p_h\end{aligned} \tag{20.98}$$

This statement represents the Jacobi theorem.

In order to prove this theorem, the derivative with respect to time of a relation from (20.98) e.g. $\frac{\partial S}{\partial a_i} = b_i$, will be determined:

$$\frac{\partial^2 S}{\partial t \partial a_i} + \sum_{k=1}^{h} \frac{\partial^2 S}{\partial q_k \partial a_i} \dot{q}_k = 0; \quad i = 1,...,h. \tag{20.99}$$

Since $S= S(q_1 ,...,q_h, a_1 ,...,a_h,t)+ a_{h+1}$ is a complete integral of (20.96)it follows that (20.96) is identically satisfied by this solution. Once this replacement is made, the derivative with respect to the constant $a_i$ will provide:

$$\frac{\partial^2 S}{\partial a_i \partial t} + \sum_{k=1}^{h} \frac{\partial H}{\partial p_k} \frac{\partial^2 S}{\partial q_k \partial a_i} = 0; \qquad i = 1,...,h \tag{20.100}$$

Subtracting the relations (20.99) and (20.100) it results:

$$\sum_{k=1}^{h} \frac{\partial^2 S}{\partial q_k \partial a_i} \left( \dot{q}_k - \frac{\partial H}{\partial p_k} \right) = 0; \qquad i = 1,...,h \tag{20.101}$$

These equations can be seen as a homogeneous algebraic system with $h$ unknowns $\left( \dot{q}_k - \frac{\partial H}{\partial p_k} \right)$. Since the determinant of this system is not zero (see (20.97)), it follows that $\left( \dot{q}_k - \frac{\partial H}{\partial p_k} \right) = 0;\ k = 1,...,h$ which corresponds to:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}; \quad i = 1,...,h, \tag{20.102}$$

the first group of Hamilton's canonical equations.
Taking now the derivative with respect to time of the second group of equalities from (20.98), e.g. $\frac{\partial S}{\partial q_i} = p_i$ it follows:

$$\frac{\partial^2 S}{\partial t \partial q_i} + \sum_{k=1}^{h} \frac{\partial^2 S}{\partial q_k \partial q_i} \dot{q}_k = \dot{p}_i; \quad i = 1,...,h \tag{20.103}$$

Replacing the solution $S= S(q_1 ,...,q_h, a_1 ,...,a_h,t)+ a_{h+1}$ in (20.96) and taking the derivative with respect to $q_i$ it follows that:

$$\frac{\partial^2 S}{\partial q_i \partial t} + \sum_{k=1}^{h} \frac{\partial H}{\partial p_k} \frac{\partial^2 S}{\partial q_k \partial q_i} + \frac{\partial H}{\partial q_i} = 0; \quad i = 1,...,h \tag{20.104}$$

Subtracting the relations (20.103) and (20.104) it follows:

$$\sum_{k=1}^{h} \frac{\partial^2 S}{\partial q_k \partial q_i} \left( \dot{q}_k - \frac{\partial H}{\partial p_k} \right) - \frac{\partial H}{\partial q_i} = \dot{p}_i; \quad i = 1,...,h \tag{20.105}$$

Using (20.102) to cancel the sum, it follows :

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i; \quad i = 1,...,h \tag{20.106}$$

which represents the second group of canonical equations. The Jacobi theorem is thus proved.

Example.
Write the Hamilton - Jacobi partial differential equation for a plane motion of a material point of mass *m* acted by another point of mass *M* by a Newtonian force of universal attraction of constant *f* and deduce the expressions of generalized coordinates and generalized momentums.

It has been deduced (20.32) the Hamilton function in this case:

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{fmM}{r} \tag{20.107}$$

The Hamilton - Jacobi partial differential equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right] - f\frac{mM}{r} = 0 \tag{20.108}$$

In order to find a complete integral, it will be tested a solution of the form

$$S = R(r) + \Theta(\theta) + T(t). \tag{20.109}$$

Substituting this solution in (20.108), it follows:

$$T' + \frac{1}{2m}(R')^2 + \frac{1}{2mr^2}(\Theta')^2 - f\frac{mM}{r} = 0 \tag{20.110}$$

Denoting by $T' = a_1$ and $Q' = a_2$, the last equation becomes:

$$r^2(R')^2 - 2fm^2Mr + 2ma_1r^2 = -a_2^2 \tag{20.111}$$

or

$$R' = \frac{1}{r}\sqrt{2fm^2Mr - 2ma_1r^2 - a_2^2} \tag{20.112}$$

The complete integral will be

$$S = a_1t + a_2\theta + \int_{r_0}^{r}\frac{1}{r}\sqrt{2fm^2Mr - 2ma_1r^2 - a_2^2}\,dr \tag{20.113}$$

The Hamilton - Jacobi partial differential equations are:

$$\frac{\partial S}{\partial a_1} = b_1, \quad \frac{\partial S}{\partial a_2} = b_2; \quad \frac{\partial S}{\partial r} = p_r, \quad \frac{\partial S}{\partial \theta} = p_\theta \tag{20.114}$$

From these, it can be deduced:

$$
\begin{aligned}
&b_1 = t + \int_{r_0}^{r} \frac{-mr}{\sqrt{2fm^2Mr - 2ma_1r^2 - a_2^2}} dr;\\
&b_2 = \theta + \int_{r_0}^{r} \frac{-a_2}{r\sqrt{2fm^2Mr - 2ma_1r^2 - a_2^2}} dr;\\
&p_r = \frac{m}{r}\sqrt{2fm^2Mr - 2ma_1r^2 - a_2^2};\\
&p_\theta = a_2.
\end{aligned}
\tag{20.115}
$$

# ANNEX. ELEMENTS OF THE THEORY OF LINE SETS

### *Definition of a line by homogeneous coordinates (Plüker coordinates)*

The unit vector $\bar{u}$ of a line (Δ) is considered with the origin in a point of coordinates *(x,y,z)*. The projections of $\bar{u}$ on the axis of a Cartesian coordinates system are:

$$\begin{aligned} a &= \cos\alpha; \quad l = y\cos\gamma - z\cos\beta \\ b &= \cos\beta; \quad m = z\cos\alpha - x\cos\gamma \\ c &= \cos\gamma; \quad n = x\cos\beta - y\cos\alpha \end{aligned} \tag{1}$$

for which the following relation is verified:

$$al + bm + cn = 0. \tag{2}$$

If instead of *a, b, c, l, m, n* are considered the scalars:

$$\lambda a, \lambda b, \lambda c, \lambda l, \lambda m, \lambda n, \tag{3}$$

with λ an arbitrary parameter, these scalars are characterizing an arbitrary vector placed on (Δ). The six scalars (3) represent the **homogeneous coordinates** (Plüker coordinates). In the following $\lambda = 1$ will be considered, so that the coordinates are those from (1).

### *Sets of lines*

For a set of *N* lines in space the homogeneous coordinates can be written in the matrix:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_N \\ b_1 & b_2 & \dots & b_N \\ c_1 & c_2 & \dots & c_N \\ l_1 & l_2 & \dots & l_N \\ m_1 & m_2 & \dots & m_N \\ n_1 & n_2 & \dots & n_N \end{bmatrix} \tag{4}$$

The following cases can be deduced from the rank *r* of the matrix:

a) r = 6 ⇒ the lines are **arbitrary** in space.
b) r = 5 ⇒ the lines are forming a **linear complex**. For N=5 this means there is a linear combination between the homogeneous coordinates $a_i$, $b_i$, $c_i$, $l_i$, $m_i$, $n_i$, $i$=1,..., N such that:

$$La_i + Mb_i + Nc_i + Al_i + Bm_i + Cn_i = 0. \tag{5}$$

In general, a linear complex is formed by the set of lines about which the resultant moment of a set of sliding vectors is zero. Particular cases are the set of lines crossing a given line or the set of lines parallels with a given plane.

c) r = 4 $\Rightarrow$ the lines are forming a **linear congruence**. For N ≥ 4 this means that there are two linear combinations (5) between the homogeneous coordinates $a_i, b_i, c_i, l_i, m_i, n_i$, $i = 1,..., N$. In general the lines of a linear congruence are the common lines of two linear complexes. A particular case is if lines are crossing two given lines.

d) r = 3 $\Rightarrow$ the lines are forming a **series of lines**. For N ≥ 3 this means that there are three linear combinations (5) between the homogeneous coordinates $a_i, b_i, c_i, l_i, m_i, n_i$, $i = 1,..., N$. In general, the lines of a series of lines are the common lines of three linear complexes. Particular cases are the lines intersecting in point in space or the lines parallel with a given line.

e) r = 2 $\Rightarrow$ the lines are forming a **planar set**. For N ≥ 2 this means that there are four linear combinations (5) between the homogeneous coordinates $a_i, b_i, c_i, l_i, m_i, n_i$, $i = 1,..., N$. Examples are the parallel lines in a plane or the concurrent lines in a plane.

f) r = 1 $\Rightarrow$ the lines are **superposed**. For N ≥ 1 this means that there are five linear combinations (5) between the homogeneous coordinates $a_i, b_i, c_i, l_i, m_i, n_i$, $i = 1,..., N$.

### *The moment of a vector about an axis in homogeneous coordinates*

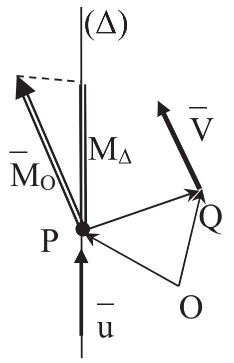


Fig. A 1.

Let be an axis (Δ) of unit vector $\bar{u}$ and a vector $\bar{V}$. Denoting by (*a, b, c, l, m, n*) the homogeneous coordinates of the unit vector $\bar{u}$ and by (*A, B, C, L, M, N*) the homogeneous coordinates of the vector $\bar{V}$ (Fig. A 1), the moment of $\bar{V}$ with respect to (Δ) can be deduced:

$$\begin{aligned} M_\Delta &= \left(\overline{PQ},\bar{V},\bar{u}\right) = \left(\overline{OQ}-\overline{OP},\bar{V},\bar{u}\right) = \left(\overline{OQ},\bar{V},\bar{u}\right) + \left(-\overline{OP},\bar{V},\bar{u}\right) \\ &= \left(\overline{OQ},\bar{V},\bar{u}\right) + \left(\overline{OP},\bar{u},\bar{V}\right) = \left(\overline{OQ}\times\bar{V}\right)\cdot\bar{u} + \bar{V}\cdot\left(\overline{OP}\times\bar{u}\right) \end{aligned} \tag{6}$$

The projections of $\bar{u}$ and $\overline{OP}\times\bar{u}$ are *a, b, c* and respectively *l, m, n* and those of $\bar{V}$ and $\overline{OQ}\times\bar{V}$ are *A, B, C*, and respectively *L, M, N*. It can be deduced for $M_\Delta$:

$$M_\Delta = La + Mb + Nc + Al + Bm + Cn. \tag{7}$$

Remark. Expression (7) can be used in computing the resultant moment of a system of sliding vectors.

Indeed, denoting by *A, B, C* the projections of the resultant vector $\bar{R}$ and by *L, M, N,* the projections of the resultant moment vector $\bar{M}_O$ one gets:

$$\bar{M}_P = \bar{M}_O + \overline{PO}\times\bar{R}. \tag{8}$$

and

$$\begin{aligned} M_\Delta &= \bar{M}_P\cdot\bar{u} = \left(\bar{M}_O + \overline{PO}\times\bar{R}\right)\cdot\bar{u} = \bar{M}_O\cdot\bar{u} + \left(\overline{PO}\times\bar{R}\right)\cdot\bar{u} \\ &= \bar{M}_O\cdot\bar{u} + \bar{R}\cdot\left(\bar{u}\times\overline{PO}\right) = \bar{M}_O\cdot\bar{u} + \bar{R}\cdot\left(\overline{OP}\times\bar{u}\right) = La + Mb + Nc + Al + Bm + Cn. \end{aligned} \tag{9}$$

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